Skepthical review: *Anomalous Transport and Velocity Statistics of Tracers in 3D Quenched Vortex Filament Fields*

Summary

This manuscript studies passive tracer transport in a *quenched*, three-dimensional velocity field generated by a random set of straight, (nominally) infinitely long vortex filaments, with velocities given by Biot–Savart superposition (Sec. 2.1). Using numerical tracer integration, the authors analyze time-averaged MSD/TAMSD scaling, velocity PDFs, velocity autocorrelations and persistence length, displacement PDFs, and a flow-topology/trapping picture based on an Okubo–Weiss-like diagnostic (Secs. 2.2–2.4, 3.1–3.6). A Lévy-walk simulation is included as a conceptual comparator for memoryless superdiffusion (Secs. 2.1–2.2, 3.2, 3.4). The central narrative is that quenched spatial disorder and coherent flow topology yield strongly superdiffusive transport with heavy-tailed velocity/displacement statistics and indications of weak ergodicity breaking (Secs. 3–4). The topic is interesting and the multi-diagnostic approach is promising, but several core elements currently prevent the results from being fully reproducible and the main universality/ergodicity claims from being quantitatively supported at the reported sample sizes and observation times.

Strengths

- Clear physical question: how Holtsmark/Lévy-type statistics and superdiffusive transport emerge (or fail to) in a deterministic *quenched* 3D filament field at finite density and finite time (Secs. 1, 3, 4).
- Multi-pronged diagnostics (TAMSD, VACF, velocity and displacement PDFs, persistence length, residence times, topology) provide a richer picture than any single exponent fit (Secs. 2.2–2.4, 3.1–3.6).
- The mechanistic attempt to link "trapping" to rotation-dominated regions via an Okubo–Weiss-type criterion is a valuable step beyond phenomenology (Secs. 2.3, 3.5).
- The quenched-disorder angle (trajectory-to-trajectory variability, long-lived correlations) is conceptually appropriate and potentially novel in this 3D filament context (Secs. 3.1, 3.4, 4).

Major issues

1. **The vortex-filament model and numerical implementation are under-specified, limiting reproducibility and making it difficult to interpret heavy tails and scaling (Sec. 2.1).** Key missing/ambiguous items include: domain geometry/size and boundary conditions; how "infinitely long" straight filaments are implemented together with periodicity (minimum-image, replicated images, Ewald-like sum, or something else); the explicit Biot–Savart expression used; filament circulation

strength(s) Γ and whether they vary across filaments; and, critically, the treatment of the Biot–Savart $1/r$ singularity (core radius / regularization / velocity cap). Trajectory integration details are also insufficient (integrator type, adaptive vs fixed step, tolerances, convergence tests). The definition and computation of the reported mean inter-filament spacing ℓ_{inter} (Table 1) is not clearly described.

Recommendation: Expand Sec. 2.1 into a fully parameterized, reproducible specification: (i) state domain size/shape and boundary conditions; (ii) write the exact single-filament velocity field used (including Γ) and how multiple filaments are superposed; (iii) describe the singularity handling (core model, cutoff radius, any maximum velocity), since it controls whether moments (e.g., variance) exist and can strongly affect tail statistics; (iv) explain precisely how periodicity is treated with "infinite" lines; (v) report the ODE integrator and step-size control, and include a simple convergence check showing statistics (e.g., TAMSD slope, high-velocity tail) are stable to Δt/tolerances; and (vi) define ℓ_{inter} explicitly (analytic from N/V or nearest-neighbor statistic) and how Table 1 values are obtained.

2. **Statistical power is currently too limited for several central claims (Secs. 3.1–3.6): only ~ 5 tracer trajectories per configuration and a short total time horizon ($T \approx 25$ s) while reported correlation times are $O(10\,\text{s})$ (Sec. 3.4, Table 1).** This makes exponent estimates ($\alpha(N)$), tail claims (velocity, residence-time, displacement PDFs), and statements about weak ergodicity breaking highly sensitive to finite-time and finite-sample effects. Reported errors such as $\alpha = 1.999 \pm 0.000$ (Table 1) are not credible as uncertainties on the underlying process; they likely reflect only regression fit residuals rather than inter-trajectory/inter-realization variability.

   Recommendation: If feasible, increase (a) the number of tracers per N and (b) the number of independent quenched filament realizations per N (both matter for quenched disorder), and extend T so that $T \gg \tau_c$ (ideally an order of magnitude). Recompute all reported exponents and tail statistics with uncertainty estimates that reflect variability across trajectories and disorder realizations (bootstrap or hierarchical resampling). If resource-limited, reframe the paper explicitly as a finite-time/finite-ensemble study: add clear error bars/bands, report how results change with T, and soften asymptotic/universality/ergodicity-breaking language in Sec. 4 accordingly.

3. **Holtsmark-related universality claims are mostly qualitative and partly conceptually inconsistent as currently presented (Secs. 3.2–3.3, 3.5–3.6, 4).** The manuscript references Holtsmark tails (e.g., $P(v) \sim v^{-5/2}$) and an expected transport exponent near $3/2$, but (i) the highest-density case still yields $\alpha \approx 1.84$ (Table 1), far from 1.5 without a finite-time scaling argument; (ii) tail "consistency" is asserted without systematic fitting ranges, uncertainties, or goodness-of-fit tests; and

(iii) the text at points cites the Holtsmark tail for velocity *components* but compares it to the *speed* PDF, which need not have the same tail form/prefactor and should be compared like-for-like in 3D.

Recommendation: Make the theory-to-measurement comparison precise and quantitative: (i) state clearly whether the theoretical $v^{-5/2}$ prediction applies to a velocity component, the speed $|v|$, or both under your assumptions; then plot/analyze the matching quantity (e.g., component PDFs if that is the theory). (ii) For each purported power law (velocity tails, residence times), perform tail fits with reported fit windows and uncertainty (e.g., MLE-based power-law fitting with goodness-of-fit and comparison to alternatives such as truncated power law/stretched exponential). (iii) For the MSD/TAMSD exponent, show local slopes $d\log(\mathrm{TAMSD})/d\log(\Delta t)$ vs Δt (with uncertainty bands) and discuss finite-time drift; if claiming convergence toward 1.5, provide supporting scaling with N and/or T or clearly label the conclusion as "Holtsmark-like/suggestive" rather than universal.

4. **The manuscript uses second-moment-based normalizations (e.g., v_c as a speed standard deviation; VACF normalization with $\langle |v|^2 \rangle$; persistence length $L_p = v_c \int C_v$) while simultaneously invoking Holtsmark-like tails that would render second moments divergent in the idealized limit (Secs. 2.2, 3.3–3.4).** Even if the simulation has an implicit cutoff (core radius, finite resolution), this must be stated because it controls these normalizations and any "collapse" claims.

   Recommendation: Explicitly identify the effective high-velocity regularization/cutoff (core model, maximum speed, grid resolution) and discuss its implications for moment existence. Consider adopting robust velocity scales (median, interquartile range, MAD) for normalization in Sec. 3.3 and for defining v_c in Sec. 2.2, or explicitly state that moments are computed for the *regularized* field and are cutoff-dependent. Ensure VACF normalization and persistence-length definitions are consistent with the intended asymptotic regime (or with the regularized model actually simulated).

5. **The Lévy-walk "ground truth" comparator is under-specified and appears internally inconsistent with standard theory (Secs. 2.1–2.2, 3.2, 3.4; Table 2).** Large discrepancies between α_{theory} and α_{measured} are not explained, and Table 2 contains at least one formula inconsistency ($\beta = 2.5$ row). As written, this weakens the intended contrast between quenched deterministic dynamics and renewal superdiffusion and may confuse readers about numerical correctness.

   Recommendation: Provide a complete Lévy-walk model specification (flight-time distribution with lower/upper cutoffs, constant speed value or speed distribution, 3D isotropic direction sampling, number of trajectories, total time, discretization) in Sec. 2.1–2.2. Validate the implementation by demonstrating recovery of known scaling over increasing T and ensemble size, or (if finite-time bias is the point) quantify the bias systematically versus T and cutoffs. Correct Table 2's theoretical-exponent row(s) and state clearly which theoretical regime/formula applies for each β.

6. **The Okubo–Weiss (OW)-like topology analysis in 3D needs clearer definition and justification (Secs. 2.3, 3.5).** OW is standard in 2D; in 3D the choice of criterion is not unique (often Q-criterion/invariants of $\nabla \boldsymbol{v}$ are used). The manuscript defines $\mathrm{OW} = \boldsymbol{s}^2 - \boldsymbol{\omega}^2$ (Eq. (3)) but does not provide explicit definitions of $\boldsymbol{s}^2$ and $\boldsymbol{\omega}^2$ (tensor contractions, prefactors), nor does it explain how $\nabla \boldsymbol{v}$ is computed robustly for a singular/regularized Biot–Savart field. Additionally, the detailed OW/trapping analysis is largely shown for $N = 40$, limiting generality.

   Recommendation: In Sec. 2.3, write explicit formulas for $\boldsymbol{s}^2$ and $\boldsymbol{\omega}^2$ in terms of the symmetric/antisymmetric parts of $\nabla \boldsymbol{v}$, including prefactors, so sign conventions are unambiguous. Justify why this OW-like diagnostic is appropriate in your 3D setting or switch to (or cross-check with) a standard 3D diagnostic (e.g., Q-criterion). Describe how $\nabla \boldsymbol{v}$ is computed (analytic differentiation vs finite differences), including grid spacing/smoothing/interpolation along trajectories. In Sec. 3.5, extend at least key OW–speed/trapping correlations to additional N (e.g., $N = 10$, 20) or clearly scope the mechanistic claim to the single case analyzed.

7. **Figure/table reporting is often missing essential metadata (sample sizes, averaging operators, normalization choices, fit ranges) and in places captions/labels appear inconsistent with plotted content (multiple figures noted in the structured report, e.g., Figs. 1–4, 6–7).** This prevents readers from verifying claims and reproducing plots.

   Recommendation: Audit all figures and captions: (i) ensure numbering matches text references; (ii) label axes with units and define all normalizations (e.g., what exactly is v_c); (iii) state N_{tracers}, number of filament realizations, trajectory duration T, binning/KDE choices, and how averages are performed (time vs ensemble); (iv) overlay and annotate reference slopes/lines used for scaling claims; and (v) add uncertainty visualization where fits are discussed (bootstrap bands/error bars), especially in tail regions.

Minor issues

1. TAMSD/MSD fitting methodology is not fully specified (Secs. 2.2, 3.2): fitting windows in Δt are not stated, it is unclear whether exponents are fit per-trajectory then averaged or fit after averaging, and uncertainty estimation procedures are not described—contributing to unrealistically small reported errors (Table 1).

   Recommendation: In Sec. 2.2, specify the exact lag-time range used for each fit and the fitting protocol (per-trajectory vs pooled/averaged). Report local-slope curves and provide uncertainty via bootstrap over trajectories (and over disorder realizations if available). Consider reporting both TAMSD and ensemble-averaged MSD to clarify finite-time/ergodicity-related differences.

2. The evidence for weak ergodicity breaking is currently suggestive but not quantitatively established (Secs. 3.1, 4). Variability across a handful of TAMSD curves is interpreted as weak ergodicity breaking without a standard metric and without analyzing dependence on T.

   Recommendation: Define and compute a standard ergodicity-breaking metric (e.g., $\mathrm{EB}(\Delta t)$ as the relative variance of TAMSD across trajectories) and show its dependence on Δt and T for each N. Compare EB behavior to the Lévy-walk baseline (once validated) to sharpen the quenched-vs-renewal contrast. If data remain limited, rephrase conclusions as "indicative of non-ergodic-like behavior at accessible times."

3. Trapping/residence-time statistics depend on an ad hoc speed threshold (25th percentile) and the analysis does not test robustness (Secs. 2.3, 3.5). Event-detection details (minimum duration, boundary handling, near-threshold splitting/merging) are not described.

   Recommendation: Describe the trapping-event detection algorithm in Sec. 2.3 (thresholding implementation, minimum event length relative to sampling, treatment of start/end events). Add a sensitivity study varying the percentile (e.g., 10/25/40%) and report how the inferred residence-time distribution and any fitted tail exponent changes.

4. Displacement PDF / Lévy-stable fitting is largely qualitative (Secs. 2.4, 3.6). The fitting method, parameter set, fit range, and uncertainty on α_{stable} are not provided, and comparisons to plausible alternatives (Gaussian, truncated stable) are not quantified.

   Recommendation: In Sec. 2.4, specify how displacement PDFs are constructed/normalized and how stable fits are performed (e.g., MLE, characteristic-function fit), including fit range. In Sec. 3.6, report $\alpha_{\mathrm{stable}}(\Delta t)$ with confidence intervals and a goodness-of-fit metric, and compare against Gaussian and truncated-stable fits to support claims of persistent non-Gaussianity.

5. The mechanistic interpretation of persistence length L_p versus ℓ_{inter} is plausible but currently presented as more definitive than warranted given limited supporting analysis (Sec. 3.4).

   Recommendation: Either (i) add a direct curvature/geometry diagnostic along trajectories (e.g., local curvature or distance-to-nearest-filament conditioned VACF decay) to support the proposed decorrelation mechanism, or (ii) soften causal language and present $L_p \ll \ell_{\mathrm{inter}}$ as suggestive evidence.

6. Several theoretical scalings are invoked without a compact derivation or clear applicability conditions (Holtsmark tails, connection to $\alpha = 3/2$, residence-time $\tau^{-5/2}$, Lévy-walk $\alpha(\beta)$ relations) (Secs. 1, 2, 3.3, 3.5, 4).

Recommendation: Add a short derivation sketch (or an appendix) with citations that states assumptions (independence/renewal, finite-speed constraint, truncation/core) and clarifies which predictions apply to components vs speed, and to time-averaged vs ensemble-averaged observables.

7. Notation and averaging operators are occasionally ambiguous (Eq. (2) VACF averaging; time vs ensemble averages for TAMSD/MSD; use of v_c as "standard deviation of speed" vs RMS speed) (Secs. 2.2, 3.2–3.4).

   Recommendation: Define averaging operators explicitly (time-average along a trajectory, ensemble average over tracers, disorder average over filament realizations) and use consistent notation throughout. Rename v_c to a standard symbol reflecting its definition (e.g., $\sigma_{|v|}$, v_{rms}, or a robust scale).

Very minor issues

1. Presentation/formatting polish: stray standalone numerals appear between paragraphs; subsection label "3.0.1" is unusual; minor LaTeX/spacing inconsistencies (α vs $\alpha(N)$, $N = 40$ vs $N = 40$, equation tagging) and some table symbols appear before being clearly (re)defined (Secs. 1–3; Tables 1–3).

   Recommendation: Proofread for formatting artifacts, standardize subsection numbering and equation formatting, and ensure all symbols used in tables/figures are defined at first use in the main text.

2. Affiliation line and occasional informal phrasing may not match journal style (title page; some captions).

   Recommendation: Replace with a conventional affiliation (or neutral "independent researcher") and edit captions/phrasing for a consistently formal scientific tone.

3. Accessibility/readability: some figures may be difficult to interpret in grayscale; legends/panel labels and metadata (trajectory counts, event counts, Δt) are sometimes missing.

   Recommendation: Adopt colorblind-safe palettes with distinct line styles/markers, increase font/line sizes, export in vector format, and include essential metadata in captions/legends.

Key statements and references

- **• For a random collection of infinitely long, straight vortex filaments whose individual induced velocities decay as $1/r$, the superposition of their Biot–Savart fields is theoretically predicted to produce a heavy-tailed Holtsmark velocity distribution with component tails $P(v) \sim v^{-5/2}$, corresponding to a Lévy walk universality class with an expected mean-squared-displacement exponent $\alpha = 1.5$.**

- *Reference(s):* (none)
- • **In two dimensions, advection in the flow induced by point vortices is known to generate superdiffusion governed by Cauchy–Lévy statistics, providing a lower-dimensional counterpart to the three-dimensional quenched vortex filament problem studied here.**
- *Reference(s):* (none)
- • **For a three-dimensional isotropic Lévy walk with flight times drawn from a heavy-tailed distribution $P(\tau) \sim \tau^{-(1+\beta)}$, theory predicts anomalous diffusion with MSD exponent α_{theory} that depends on β (e.g., $\alpha_{\text{theory}} = 1.8$ for $\beta = 1.2$, $\alpha_{\text{theory}} = 1.5$ for $\beta = 1.5$, $\alpha_{\text{theory}} = 1.2$ for $\beta = 1.8$, and $\alpha_{\text{theory}} = 1.0$ for $\beta = 2.5$), but finite-time numerical simulations systematically underestimate these exponents, yielding α_{measured} values of 1.655, 1.049, 1.069, and 1.012, respectively.**
- *Reference(s):* (none)

Mathematical consistency audit

This section audits **symbolic/analytic** mathematical consistency (algebra, derivations, dimensional/unit checks, definition consistency).

Maths relevance: light

The PDF contains a small set of explicit equations (TAMSD, VACF, Okubo–Weiss) and several central theoretical scaling claims (Holtsmark tails, MSD exponent, residence-time tails, Lévy-walk exponent mapping). The explicit equations are largely algebraically and dimensionally consistent, but the paper relies on second moments (standard deviation and $\langle |v|^2 \rangle$) while simultaneously asserting a $v^{-5/2}$ tail that would make those moments diverge without a stated truncation/regularization. Multiple key exponent relationships are asserted without derivations, preventing full internal verification from the document.

Checked items

1. ✔ **Tracer equation of motion** (Sec. 2.1, p.3 (text: $\dot{\mathbf{r}}(t) = \mathbf{v}(\mathbf{r}(t))$))
   - **Claim:** Tracer position evolves via advection in the steady velocity field.
   - **Checks:** definition consistency, dimensional/units
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** Velocity field is defined at tracer positions, Field is time-independent (quenched)
   - **Notes:** Standard advection ODE; dimensions consistent (L/T on both sides).
2. ✔ **TAMSD definition** (Eq. (1), Sec. 2.2, p.3)

- **Claim:** Time-averaged MSD is $(T - \Delta t)^{-1} \int_0^{T-\Delta t} |\mathbf{r}(t' + \Delta t) - \mathbf{r}(t')|^2 dt'$.
- **Checks:** algebra/structure, dimensional/units, normalization/constraints
- **Verdict:** PASS; confidence: high; impact: moderate
- **Assumptions/inputs:** $0 < \Delta t < T$, $\mathbf{r}(t)$ is well-defined and measurable over $[0, T]$
- **Notes:** Normalization and integration limits are consistent; result has units of length^2. (Discrete-time implementation not audited.)

3. ✔ **Power-law MSD scaling fit** (Sec. 2.2 and Sec. 3.2, pp.3,6)

   - **Claim:** Ensemble-averaged TAMSD scales as $\langle \delta^2(\Delta t) \rangle_t \sim (\Delta t)^\alpha$ over an intermediate range; α extracted from a log-log slope/fit.
   - **Checks:** definition consistency, notation consistency
   - **Verdict:** PASS; confidence: medium; impact: moderate
   - **Assumptions/inputs:** There exists a lag-time range with approximately constant logarithmic slope, Ensemble averaging over tracers/realizations is well-defined
   - **Notes:** Procedure is standard. Notation could better distinguish time- vs ensemble-averaging (clarity issue, not an algebraic error).

4. ✖ **VACF definition and normalization** (Eq. (2), Sec. 2.2, p.3)

   - **Claim:** $C_v(\Delta t) = \langle \mathbf{v}(t) \cdot \mathbf{v}(t + \Delta t) \rangle / \langle |\mathbf{v}(t)|^2 \rangle$.
   - **Checks:** dimensional/units, definition consistency, existence of moments
   - **Verdict:** FAIL; confidence: medium; impact: critical
   - **Assumptions/inputs:** Second moment $\langle |v|^2 \rangle$ exists (finite) under the modeled velocity statistics, Averaging operator $\langle \cdot \rangle$ is well-defined (time/ensemble)
   - **Notes:** C_v is dimensionless as written, but later the paper asserts a Holtsmark tail $P(v) \sim v^{-5/2}$. Such a tail implies a divergent $\langle |v|^2 \rangle$ for an untruncated distribution, making Eq. (2)'s normalization undefined without an explicit cutoff/regularization.

5. ✔ **Correlation time definition (first zero-crossing)** (Sec. 2.2, p.3 (text after Eq. (2)))

   - **Claim:** τ_c is defined as the first zero-crossing of $C_v(\Delta t)$.
   - **Checks:** definition consistency, edge cases
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** $C_v(\Delta t)$ crosses zero within observation time (otherwise undefined)
   - **Notes:** Definition is clear; the paper appropriately notes cases where no zero crossing occurs within the window.

6. ⚠ **Persistence length formula** (Sec. 2.2, p.3 (text: $L_p = v_c \int_0^{\tau_c} C_v(\Delta t) d\Delta t$))

- **Claim:** L_p equals a characteristic speed scale times the integrated VACF up to τ_c.
- **Checks:** dimensional/units, definition consistency, existence of moments
- **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
- **Assumptions/inputs:** v_c is a well-defined characteristic speed, Integral exists and τ_c is defined
- **Notes:** Dimensionally consistent (L/T × T = L). However, v_c is defined as the standard deviation of speed; under the stated $v^{-5/2}$ tail, that standard deviation is not defined without truncation. If the field is regularized/truncated, the formula is fine, but the truncation is not specified.

7. ✖ **Velocity scale definition $v_c(N)$ as standard deviation** (Sec. 2.2, p.3 (text in velocity statistics paragraph))

   - **Claim:** $v_c(N)$ is the standard deviation of the speed distribution and is used to normalize speeds across N.
   - **Checks:** existence of moments, definition consistency
   - **Verdict:** FAIL; confidence: high; impact: critical
   - **Assumptions/inputs:** Speed variance exists (finite) under the stated asymptotic statistics
   - **Notes:** If $P(v) \sim v^{-5/2}$ holds asymptotically without truncation, the second moment diverges and the standard deviation is undefined. This conflicts with using v_c as a mathematical scale unless an explicit cutoff/regularization is stated.

8. ⚠ **Okubo–Weiss parameter definition** (Eq. (3), Sec. 2.3, p.4)

   - **Claim:** $\mathrm{OW} = s^2 - \omega^2$; $\mathrm{OW} < 0$ indicates rotation-dominated regions and $\mathrm{OW} > 0$ indicates strain-dominated regions.
   - **Checks:** definition consistency, missing definitions
   - **Verdict:** UNCERTAIN; confidence: medium; impact: minor
   - **Assumptions/inputs:** s^2 and ω^2 are nonnegative scalars derived consistently from ∇v, Sign convention matches the stated interpretation
   - **Notes:** Without explicit formulas for s^2 and ω^2 (including any prefactors), OW's exact meaning and sign convention in 3D cannot be verified from the document.

9. ✔ **Lévy-walk flight-time tail specification** (Sec. 2.1, p.3)

   - **Claim:** Flight times are drawn from $P(\tau) \sim \tau^{-(1+\beta)}$ and direction is reset isotropically after each flight.
   - **Checks:** definition consistency, notation consistency
   - **Verdict:** PASS; confidence: medium; impact: minor

- **Assumptions/inputs:** Renewal property (independent flights), Constant speed magnitude during each flight (paper says constant velocity but does not specify magnitude distribution)
- **Notes:** Model description is internally coherent, though the magnitude of the constant velocity is not explicitly specified.

10. ⚠ **Mapping from Lévy-walk β to α_{theory} in Table 2** (Table 2, Sec. 3.0.1, p.5; definition of β in Sec. 2.1, p.3)

    - **Claim:** Given β in $P(\tau) \sim \tau^{-(1+\beta)}$, the theoretical diffusion exponent α_{theory} takes listed values (e.g., $\beta = \mathbf{1.5} \to \alpha_{\text{theory}} = \mathbf{1.5}$).
    - **Checks:** derivation completeness, definition consistency
    - **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
    - **Assumptions/inputs:** A specific analytical relationship exists between β and MSD exponent for the defined walk, Speed distribution/magnitude is fixed in the theoretical mapping
    - **Notes:** The paper does not provide the formula linking β to α_{theory}, nor the assumptions under which it holds. From the PDF alone, the entries in Table 2 cannot be analytically verified.

11. ⚠ **Holtsmark tail claim for velocity statistics** (Sec. 2.2, p.3 ($P(v) \sim v^{-5/2}$); Sec. 3.3, p.7 (components vs speed wording))

    - **Claim:** Holtsmark theory predicts a power-law velocity tail $P(v) \sim v^{-5/2}$ (stated for components) and empirical speed PDFs are consistent with it.
    - **Checks:** notation/definition consistency, missing justification
    - **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
    - **Assumptions/inputs:** Clear distinction between component marginal PDF and speed (magnitude) PDF, or justification that exponents coincide
    - **Notes:** The document mixes 'velocity components' with 'speed PDFs' while using the same exponent. This may be correct under isotropy but needs an explicit statement/derivation to be internally checkable.

12. ⚠ **Holtsmark → MSD exponent $\alpha = 1.5$ claim** (Introduction, p.2; Sec. 3.2, p.6 (reference to $\alpha = 1.5$))

    - **Claim:** Holtsmark velocity statistics imply a superdiffusive MSD exponent $\alpha = 1.5$.
    - **Checks:** derivation completeness
    - **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
    - **Assumptions/inputs:** A specified transport model linking velocity statistics to displacement scaling (including correlation assumptions)

- **Notes:** No derivation is provided in the PDF showing how the Holtsmark tail leads to $\alpha = 1.5$ for this quenched deterministic system; key assumptions are not stated, so the analytic link cannot be verified.

13. ⚠ **Holtsmark → residence-time tail $P(\tau) \sim \tau^{-5/2}$ claim** (Sec. 3.5, p.8; Fig. 5 caption, p.10)

    - **Claim:** Residence times of low-speed trapping events have a predicted tail $P(\tau) \sim \tau^{-5/2}$ derived from Holtsmark statistics.
    - **Checks:** derivation completeness, definition consistency
    - **Verdict:** UNCERTAIN; confidence: low; impact: moderate
    - **Assumptions/inputs:** A defined mapping from low-speed events to Holtsmark-distributed velocities, Specification of how trapping duration depends on local velocity magnitude/topology
    - **Notes:** The PDF does not provide the derivation; without the mechanism relating trapping duration to velocity statistics, the exponent claim cannot be audited symbolically.

14. ⚠ **Displacement PDFs and Lévy-stable index** (Sec. 3.6, p.9; Fig. 7 caption, p.11)

    - **Claim:** Short-lag displacement PDFs are described by a Lévy-stable distribution with stability index $\alpha_{\text{stable}} \approx 1.5$; cores become more Gaussian as Δt increases beyond τ_c.
    - **Checks:** notation consistency, derivation completeness (fit/model statement)
    - **Verdict:** UNCERTAIN; confidence: low; impact: minor
    - **Assumptions/inputs:** Definition of the fitted stable family and parameterization (not shown), Sufficient decorrelation beyond τ_c for CLT-type arguments to apply (heuristic)
    - **Notes:** This is largely interpretive; the analytic mapping between Holtsmark velocity statistics and displacement stable index is not derived in the document, and the fitting functional form is not specified.

Limitations

- The audit is restricted to the provided 12-page PDF text/figures; no supplemental material, appendices, or detailed derivations are available to verify exponent mappings.
- The Biot–Savart law expression for filaments and any regularization/core model are not explicitly written in the PDF text provided; this prevents checking moment existence and cutoff assumptions directly.
- Figures are referenced for qualitative behavior, but the audit does not validate any numerical fits or plotted values; only the consistency of the stated formulas/claims is assessed.

Numerical results audit

This section audits **numerical/empirical** consistency: reported metrics, experimental design, baseline comparisons, statistical evidence, leakage risks, and reproducibility.

Eight internal arithmetic/cross-reference checks passed (sampling count, implied trajectory count, Table 1 value-to-text matches, approximation sanity check, persistence-length sanity check, inequality claim, and monotonic ordering). One check failed: Table 2's α_{theory} values match $\alpha_{\text{theory}} = 3 - \beta$ for three rows but not for $\beta = 2.5$.

Checked items

1. ✔ **C1__time__points__consistency** (Page 3, Methods: trajectory duration, dt, and data points)

   - **Claim:** "tracked for a total duration of approximately 25 seconds, with positions recorded at a time step of $dt = 0.05$ s, yielding 500 data points per trajectory."
   - **Checks:** sampling__count__consistency
   - **Verdict:** PASS
   - **Notes:** Computed $T/dt = 25/0.05 = 500$ matches reported 500 within off-by-one tolerance for endpoint convention.

2. ✔ **C2__total__tracer__count** (Page 3, Methods: number of fields and tracers)

   - **Claim:** "For each of the four quenched velocity fields, we simulated the trajectories of five passive tracers..." (with four N values: 5, 10, 20, 40).
   - **Checks:** count__product__consistency
   - **Verdict:** PASS
   - **Notes:** Arithmetic implication: 4 fields $\times$ 5 tracers/field = 20 total tracer trajectories implied.

3. ✔ **C3__table1__alpha__uncertainty__format** (Page 5, Table 1 (row $N = 5$) and Page 6 text)

   - **Claim:** Table 1 reports for $N = 5$: $\alpha(N) = 1.999$ and α SE $= 0.000$; text states "$\alpha = 1.999 \pm 0.000$."
   - **Checks:** cross__reference__value__match
   - **Verdict:** PASS
   - **Notes:** Table and text match exactly at printed precision for both α and its SE.

4. ✔ **C4__table1__alpha__N40__text__match** (Page 5, Table 1 (row $N = 40$) and Page 6 text)

   - **Claim:** Table 1 reports for $N = 40$: $\alpha(N) = 1.835$ and α SE $= 0.003$; text states "$\alpha = 1.835 \pm 0.003$ for $N = 40$."

- **Checks:** cross_reference_value_match
- **Verdict:** PASS
- **Notes:** Table and text match exactly at printed precision for both α and its SE.

5. ✔ **C5_table1_tau_c_approx_10s_claim** (Page 7, Section 3.4 text and Page 5, Table 1)

   - **Claim:** "τ_c is approximately 10 s for $N = 10$ and $N = 40$." Table 1: $\tau_c = 10.17$ s ($N = 10$) and 9.555 s ($N = 40$).
   - **Checks:** approximation_consistency
   - **Verdict:** PASS
   - **Notes:** Relative deviations vs 10 s: $N = 10$ is 0.017, $N = 40$ is 0.0445; both within the heuristic 10% tolerance for "approximately".

6. ✔ **C6_persistence_length_definition_vs_table** (Page 3 Eq. (2) paragraph (Lp definition) and Page 5 Table 1)

   - **Claim:** Persistence length defined as $L_p = v_c \int_0^{\tau_c} C_v(\Delta t) d\Delta t$; Table 1 reports $L_p = 0.313$ m for $N = 10$ and $L_p = 0.578$ m for $N = 40$ with $\tau_c = 10.17$ s and 9.555 s, respectively.
   - **Checks:** derived_quantity_sanity_bounds
   - **Verdict:** PASS
   - **Notes:** Sanity computation L_p/τ_c: $N = 10 \to 0.03078\,\mathrm{m/s}$; $N = 40 \to 0.06049\,\mathrm{m/s}$; non-negative and ratio ≈ 1.97 (not extreme). Not a proof without v_c or $C_v(\Delta t)$ data.

7. ✔ **C7_Lp_vs_linter_claim** (Page 8, Section 3.4 text and Page 5 Table 1)

   - **Claim:** "The persistence length L_p (Table 1) is substantially smaller than the mean inter-filament spacing ℓ_{inter}." Table 1 provides both where available.
   - **Checks:** inequality_check
   - **Verdict:** PASS
   - **Notes:** Inequalities hold: $N = 10$ and $N = 40$ both have $L_p < \ell_{\text{inter}}$. Ratios: $N = 10$ $L_p/\ell_{\text{inter}} \approx 0.0337$; $N = 40 \approx 0.0988$.

8. ✔ **C8_linter_monotonic_decrease_with_N** (Page 5, Table 1 (ℓ_{inter} column))

   - **Claim:** Mean inter-filament spacing ℓ_{inter} values: 11.70 ($N = 5$), 9.28 ($N = 10$), 7.37 ($N = 20$), 5.85 ($N = 40$).
   - **Checks:** monotonicity_check
   - **Verdict:** PASS
   - **Notes:** Sequence is strictly decreasing with N: $11.70 > 9.28 > 7.37 > 5.85$.

9. ✖ **C9_table2_alpha_theory_relation** (Page 5, Table 2 (Lévy walk ground truth))

- **Claim:** Table 2 lists $(\beta, \alpha_{\text{theory}})$: $(1.2, 1.8)$, $(1.5, 1.5)$, $(1.8, 1.2)$, $(2.5, 1.0)$.
- **Checks:** internal_pattern_check
- **Verdict:** FAIL
- **Notes:** For rows 1–3, α_{theory} equals $3 - \beta$ exactly. For $\beta = 2.5$, expected $3 - \beta = 0.5$ but table lists $\alpha_{\text{theory}} = 1.0$ (abs diff 0.5; rel diff 0.5).

Limitations

- Only parsed text was available; no machine-readable tables beyond the text dump, and no underlying simulation data arrays were provided.
- Checks requiring re-fitting distributions/exponents from plotted curves are not feasible without numeric data (and plot-value extraction is out of scope).
- Several claims are qualitative ("approximately", "consistent with") and can only be sanity-checked against reported summary numbers, not rigorously verified.