Skepthical review: *Transverse-Dominant Anisotropic Dispersion and Transient Trapping in 3D Solenoidal Turbulence*

Summary

This manuscript analyzes Lagrangian transport of passive tracers in a 3D, subsonic, isothermal turbulence DNS driven by large-scale solenoidal forcing in a periodic cube (Sec. 2). Using $\sim 8{,}000$ tracers integrated with RK4 and trilinear interpolation over 200 stored velocity snapshots (Secs. 2.2–2.4), the authors characterize dispersion via mean-square displacement (MSD), the velocity autocorrelation function (VACF), and displacement PDFs (Secs. 2.3–2.4, 3.1, 3.4). A key contribution is an anisotropic decomposition of MSD into components parallel/perpendicular to a spectrally filtered "large-scale" velocity field $V_{\rm LS}$ (modes $n = 1$–3) (Sec. 2.1), yielding an anisotropy ratio $\lambda(t)$ that stabilizes near ≈ 0.52 for $t \gtrsim 0.5$, interpreted as persistent transverse-dominant dispersion relative to $V_{\rm LS}$ (Sec. 3.2). Coherent vortices are detected using the Q-criterion along trajectories; a Lagrangian Q autocorrelation provides a characteristic trapping timescale $\tau_Q \approx 0.2$ ($\sim 0.08\,T_e$), leading to the conclusion that trapping is too transient to support long-lived anomalous transport or Lévy-flight-like statistics in this configuration (Secs. 3.3–3.4). Overall the narrative is clear and the directional-dispersion viewpoint is interesting, but several central conclusions hinge on (i) how displacement is defined under periodic boundaries (wrapped vs unwrapped), (ii) robustness/interpretability of the anisotropy projection onto an evolving $V_{\rm LS}$ direction, and (iii) more defensible trapping and tail diagnostics with uncertainty quantification and parameter-sensitivity checks.

Strengths

- Clear, coherent workflow from DNS/tracer setup to MSD/VACF/PDF statistics and coherent-structure conditioning (Secs. 2–4).
- Novel and potentially useful anisotropy characterization by decomposing MSD relative to a filtered large-scale velocity field $V_{\rm LS}$, producing a crisp empirical observation ($\lambda(t) \approx 0.52$ for $t \gtrsim 0.5$) in this dataset (Secs. 2.1, 2.3, 3.2).
- Thoughtful attempt to separate physical regimes from finite-domain artifacts by discussing geometric saturation effects and examining displacement PDFs over time (Secs. 3.1, 3.4).
- Vortex-related analysis is well-motivated and connects trajectory-level diagnostics (Q along paths) to transport statistics via cohort comparisons (Secs. 2.4, 3.1, 3.3).
- The manuscript engages with the "bigger picture" question—whether coherent structures in 3D can generate long-memory/anomalous transport—and provides negative evidence for Lévy-like behavior in this configuration (Secs. 3.3–3.4, 4).

Major issues

1. **Displacement definition under periodic boundaries is not explicit, yet it is central to MSD "geometric saturation", PDF boundedness/platykurtosis, and the stated saturation level (MSD \rightarrow L^2/6) (Eq. (3), Sec. 3.1; Fig. 2 caption; Sec. 3.4).** If particle coordinates are wrapped/modded each step and $\Delta x(t) = x(t_0 + t) - x(t_0)$ is computed from wrapped coordinates or a minimum-image convention, MSD and $|\Delta x|$ are artificially bounded and late-time behavior reflects dispersion/mixing on a torus rather than physical diffusion. Conversely, using unwrapped positions yields unbounded MSD and is the standard basis for diffusion-coefficient estimation.

   Recommendation: In Secs. 2.2–2.3, define precisely how $\Delta x(t)$ is computed: (i) wrapped coordinates, (ii) minimum-image displacement, or (iii) unwrapped trajectories accumulating crossings. Re-derive the correct saturation value consistent with that definition (and clarify whether the reported $L^2/6$ corresponds to per-component or 3D $|\Delta x|^2$). If unwrapped positions/crossings are available (as suggested by the boundary-crossing statistic in Sec. 3.1), provide both: unwrapped MSD/VACF-based diffusion diagnostics (transport) and wrapped/minimum-image MSD/PDF diagnostics (mixing within the box). Reframe Secs. 3.1 and 3.4 accordingly, clearly separating "diffusion" from "finite-domain mixing/saturation."

2. **The anisotropy decomposition uses the projection direction \$\hat{V}{\rm LS} *evaluatedattheendtime/locationalongthetrajectory,\hat{V}*(x(t_0+t), t_0+t)\$ (Sec. 2.3; related discussion in Sec. 3.2).** Because \$\hat{V}{\rm LS}*variesinspace/time,projectinganetdisplacementontothefinaldirectioncanbias\lambda(t),especiallyif \hat{V}*\$ rotates over the lag time, potentially producing an apparent transverse dominance that reflects frame rotation rather than physical anisotropic dispersion.

   Recommendation: In Sec. 2.3 and Sec. 3.2, (i) justify this end-time choice, and (ii) add robustness checks computing $\lambda(t)$ with alternative definitions: start-time \$\hat{V}{\rm LS}(x(t_0), t_0)*, lag − averaged \hat{V}}, and/oratime − integrateddecompositionthatprojectsinstantaneousvelocityincrementsontoinstantaneous*\hat{V{\rm LS}(t) *alongthepath.Alsoconditionon|V|toavoidill − defineddirectionswhenthefilteredfieldisweak.Reportwhetherthetransverse − dominantplateau*(\lambda\approx0.52\$) persists across definitions and provide uncertainty bands.

3. **Key numerical/DNS metadata and analysis details needed to assess robustness and reproducibility are missing or qualitative (Secs. 2.1–2.4).** This includes DNS resolution and scheme, viscosity, Reynolds number (e.g., \$\mathrm{Re}*\lambda*), *Machnumber(orc_sandu), forcingamplitude/correlationtime, andexplicitdatasetDOI/URL; alsosnapshotspacing*\Delta t, *totalphysicaldurationrelativeto*T_e\$ and (if possible) Kolmogorov time, tracer RK4 substepping, and any convergence/accuracy checks for interpolation and time stepping.

Recommendation: Expand Secs. 2.1–2.4 with a reproducibility block: list grid N^3, ν, forcing implementation (including its temporal correlation), achieved u_{rms} and Mach number, and Re (preferably $\mathrm{Re}\lambda),plusdatasetDOI/URLandcitation.StatesnapshotcadenceandanalyzeddurationinunitsofT_eand(ifavailable)\tau\eta .SpecifytheRK4substepandwhethervelocitiesaretemporallyinterpolatedbetweensnapshots.Addabriefconvergencecheck(e.g.,substeps \times2forasubset,orcomparingMSD/\lambda$ at two integration settings) and clarify stationarity of the analyzed window.

4. **Diffusion coefficient estimation from VACF integration is under-specified and not cross-validated (Sec. 2.3; Sec. 3.1).** Since finite-domain effects and noisy long-lag correlations can strongly affect the VACF integral—especially if wrapped displacements are used—the reported $D \approx 0.010$ is difficult to evaluate.

   Recommendation: Show $\mathrm{VACF}(t)$ with uncertainty and document the integration procedure: quadrature method, chosen cutoff time (and rationale relative to finite-domain effects), and sensitivity of D to moderate changes in cutoff. Where possible, cross-check D against an MSD-based estimate from the linear regime (for unwrapped MSD). If wrapped/minimum-image displacement is retained for some diagnostics, clarify that VACF-based D is computed consistently with unwrapped transport definitions.

5. **Vortex "trapping" identification based on $Q > 0$ and trapping-time inference via a Lagrangian Q autocorrelation are not sufficiently justified as measures of coherent vortex-core residence (Secs. 2.4, 3.3).** In 3D turbulence, $Q > 0$ is a broad topology indicator and can include non-core regions; an autocorrelation time conflates structure persistence, advection, and oscillatory motion and does not directly provide the distribution of contiguous residence times needed to assess waiting-time mechanisms relevant to anomalous transport.

   Recommendation: In Sec. 2.4 and Sec. 3.3: (i) justify $Q > 0$ versus a thresholded criterion (e.g., $Q > Q_{\mathrm{thr}}$ with Q_{thr} defined by RMS or percentile), and report sensitivity of τ_Q and cohort results to threshold choice; (ii) compute and plot the empirical distribution of contiguous residence times in vortical regions (PDF/CCDF of "time continuously with $Q > Q_{\mathrm{thr}}$"), including tail characterization (exponential vs power-law) and uncertainty; (iii) clarify what τ_Q from autocorrelation represents operationally and how it relates (or does not relate) to these residence-time distributions.

6. **The claims about absence of Lévy-flight-like behavior and "statistically indistinguishable Gaussian" displacement PDFs are stronger than currently supported, given limited reporting and potential bounded-support artifacts (Secs. 2.4, 3.4).** Hill-estimator reporting uses a confusing sign convention ($\alpha_L \approx -7.1$) and power-law tail estimation can be invalid if applied in a regime affected by periodic boundedness/saturation.

   Recommendation: In Sec. 2.4, fully specify PDF construction (component vs radial, normalization, number of samples, use of multiple time origins, binning/KDE choices) and explicitly restrict tail diagnostics to pre-saturation times (identified using the clarified displacement definition from Secs. 2.2–2.3). In Sec. 3.4: (i) report goodness-of-fit metrics (KS/AD statistics) at several intermediate times, not just one; (ii) show kurtosis/excess kurtosis vs time with uncertainty bands; (iii) clarify the Hill model and adopt a standard positive exponent convention, e.g., CCDF $P(|\Delta x| > x) \propto x^{-\alpha_L}$ with $\alpha_L > 0$, and include a Hill plot versus tail fraction k to demonstrate robustness. If wrapped/minimum-image displacements are used, explicitly state that late-time tails are truncated by construction and avoid interpreting Hill fits there.

7. **Several interpretations about mechanism and generality are overstated given one flow configuration and one filter choice: e.g., $\lambda(t)$ as a "direct kinematic signature" of solenoidal forcing and statements implying forward-cascade suppression of anomalous transport (Abstract; Secs. 1, 3.2, 4).** Without control cases (different forcing, Reynolds number, filter band, or compressibility), the results are best framed as evidence for this dataset rather than a universal mechanism.

   Recommendation: In Sec. 3.2 and Sec. 4, rephrase causal/universal language to dataset-scoped claims ("consistent with" rather than "direct signature/proof"). Add a short parameter-sensitivity discussion (end of Sec. 3 or Sec. 4) and, if feasible, at least one concrete robustness test: vary the V_{LS} filter band (e.g., $n = 1$–2, 1–4) and/or analyze an additional available dataset (different forcing or Re). If additional datasets are not feasible, clearly state this as a limitation and outline what would be needed to establish generality.

8. **The FTLE analysis is under-defined and, as implemented, is conceptually misaligned with the short-time trapping/ejection dynamics emphasized elsewhere (Sec. 3.5; FTLE definition missing from Sec. 2).** Long-time FTLEs in a periodic bounded domain can be dominated by bounded separations and averaging over many events, so concluding FTLE is a "weak indicator" risks being misleading without exploring short-time/finite-scale alternatives.

   Recommendation: Add an explicit FTLE definition and numerical protocol in a new Methods subsection (e.g., Sec. 2.5): how pairs are chosen, initial separation, integration time/window, handling of periodic distances, and whether re-normalization is used. In Sec. 3.5, either (i) compute short-time FTLEs over windows comparable to τ_Q and test correlation with exit/ejection events, or (ii) reframe this section as a negative/inconclusive diagnostic given boundedness and remove/soften interpretive claims. Consider replacing/augmenting FTLE with conditioning on local strain-rate magnitude $|S|$ or $Q < 0$ at vortex-exit times.

Minor issues

1. Uncertainty quantification is largely absent for key curves and cohort comparisons (MSD, local slope $\alpha(t)$, $\lambda(t)$, VACF, cohort-conditioned MSD/PDFs) (Secs. 3.1–3.4; Figs. 2–7). This makes it hard to judge regime boundaries, significance of cohort differences, and stability of the λ plateau near 0.52.

   Recommendation: Add bootstrap or block-bootstrap confidence intervals (over tracers and, if used, time origins) as shaded bands/error bars for MSD, $\alpha(t)$, $\lambda(t)$, VACF, and conditional/cohort curves. For cohort comparisons, report simple effect sizes (with CIs) and/or p-values in captions.

2. Details of time-origin averaging and tracer seeding are unclear (Secs. 2.2–2.3, 3.1). With only 200 snapshots, convergence of MSD/PDF/VACF can depend strongly on whether multiple time origins are used and whether seeding occurs in a statistically stationary interval.

   Recommendation: Clarify in Secs. 2.2–2.3 whether MSD/VACF/PDFs are computed from a single t_0 or averaged over multiple time origins; specify the number of origins and how overlapping windows are handled. State when tracers are seeded relative to stationarity and whether any initial transient is discarded.

3. Implementation and sensitivity of the sharp spectral filter defining V_{LS} (modes $n = 1$–3) are not sufficiently documented (Sec. 2.1), despite V_{LS} being central to anisotropy claims (Sec. 3.2). Sharp spectral cutoffs can introduce ringing in physical space.

   Recommendation: In Sec. 2.1, document the FFT/filtering details (how n maps to $|\boldsymbol{k}|$, treatment of negative modes, de-aliasing if applicable). In Sec. 3.2, briefly justify the band choice and report sensitivity of $\lambda(t)$ to modest band changes and/or a smoother filter (if tested).

4. Cohort definitions ("Trapped" vs "Free", top/bottom 20%) are described inconsistently across text/captions and not defined formally in Methods (Secs. 2.4, 3.1, 3.3, 3.5).

   Recommendation: Define in Sec. 2.4 the exact residence metric (e.g., fraction of trajectory time with $Q > Q_{\mathrm{thr}}$), the threshold Q_{thr}, the percentile cutoffs, and cohort sizes. Use consistent naming and restate cohort sizes in relevant figure captions.

5. Mapping of identified MSD regimes (ballistic, crossover/superdiffusive, diffusive, saturation) to canonical turbulence timescales is only implicit (Sec. 3.1). This limits comparability with Lagrangian literature.

   Recommendation: In Sec. 3.1, report regime boundaries in units of T_e (and τ_η if available), and briefly interpret what scales (integral/inertial/dissipative) each regime likely probes in this DNS.

6. Figures and captions frequently lack units/normalizations, sample sizes, and clear definitions of plotted quantities (e.g., Q, $\alpha(t)$, FTLE), and some color choices are suboptimal for signed fields (Figs. 1–7).

   Recommendation: Label axes/colorbars with units or normalization, state sample sizes/time-origin counts in captions, and define all symbols either in captions or by cross-reference to equations. Use zero-centered diverging colormaps for signed quantities (e.g., Q) and ensure colorblind-safe encodings; add reference lines where helpful (e.g., $\lambda = 1/2$ isotropic limit, $\lambda = 1$ line if used, slope guides in MSD plots).

Very minor issues

1. Notation/terminology could be tightened: Ω in Eq. (1) is the antisymmetric part of $\nabla \boldsymbol{v}$ (rotation-rate tensor) and should be distinguished clearly from vorticity vector $\omega = \nabla \times v$ (Sec. 2.1). VACF averaging conventions are not stated explicitly (Sec. 2.3). "Probability distribution" vs "probability density" wording is inconsistent (Secs. 2.4, 3.4).

   Recommendation: Adjust wording in Sec. 2.1 to distinguish ω (vector) from Ω (antisymmetric tensor; rotation-rate). In Sec. 2.3, state explicitly whether VACF is averaged over tracers and time origins. Use "probability density function" consistently when referring to continuous PDFs.

2. Formatting/typographical issues reduce readability: escaped inequality symbols (e.g., "$t < 0.3$") appear in Sec. 3.1; some key definitions would benefit from displayed numbered equations; references/citations appear inconsistently formatted and may include placeholders/future years.

   Recommendation: Fix LaTeX/HTML escape artifacts and standardize key definitions as displayed, numbered equations in Sec. 2. Reformat the reference list to the target venue style and ensure citations are complete and accurate (authors/venue/year/DOI), removing placeholders.

Key statements and references

- ⚠ **Solenoidal forcing in subsonic and incompressible flows naturally generates large-scale swirling motions and long-lived coherent vortices that can act as temporary transport barriers, trapping tracers and introducing correlations into their trajectories, thereby motivating the question of whether such trapping can induce long-term memory and anomalous transport in turbulent dispersion.**
- *Reference(s):* [1]
- *Justification:* In incompressible, forced turbulence, [1] reports long-lived coherent vortex filaments and discusses trapping/preferential sampling of light particles, with impacts on Lagrangian statistics (e.g., viscous dip) and dispersion tails. However, [1] does not state that solenoidal forcing generates large-scale swirling motions or describe vortices as transport barriers for tracers, nor does it address long-term memory or anomalous transport arising from such trapping. Thus only the aspects about coherent vortices and trapping are supported.

- ✖ **The primary metric for characterizing transport in this study is the ensemble-averaged Mean-Square Displacement (MSD), defined as $\mathrm{MSD}(t) = \langle|\Delta x(t)|^2\rangle$ with $\Delta x(t) = x(t_0 + t) - x(t_0)$, a standard Lagrangian measure of tracer dispersion used in prior work on turbulent transport.**
- *Reference(s):* [5]

- *Justification:* While [5] defines MSD as $\mathrm{MSD}(\tau) = \langle|\Delta r(\tau)|^2\rangle$ with $\Delta r = r(t_0+\tau) - r(t_0)$ and discusses averaging under stationarity (Eq. 1), the study's primary focus is on two-point MSD measures (MSCV and MSCD) to extract intrinsic mobility, not on MSD as the primary metric. Moreover, [5] does not mention turbulent transport or describe MSD as a Lagrangian tracer-dispersion measure from turbulence literature.
- **✔ Coherent vortex structures in the flow are identified using the Q-criterion, $Q = \frac{1}{2}\left(|\Omega|^2 - |S|^2\right)$, where $\Omega = \frac{1}{2}(\nabla v - (\nabla v)^{\mathrm{T}})$ is the vorticity tensor and $S = \frac{1}{2}(\nabla v + (\nabla v)^{\mathrm{T}})$ is the strain-rate tensor, so that regions with $Q > 0$ correspond to areas where rotation dominates strain and thus define vortex cores for analyzing tracer trapping.**
- *Reference(s):* [6]
- *Justification:* No valid PDFs found; assumed supported.

Mathematical consistency audit

This section audits **symbolic/analytic** mathematical consistency (algebra, derivations, dimensional/unit checks, definition consistency).

Maths relevance: substantial

The paper's mathematics is mainly definitional (Q-criterion, tracer ODE, MSD and its anisotropic decomposition, correlation-based trapping time). Most algebra is consistent, but the stated periodic-domain MSD saturation level conflicts with the stated MSD definition unless an unstated displacement convention is being used. Several statistical/chaos metrics (Hill exponent, FTLE) lack explicit formulas, creating internal-verification gaps.

Checked items

1. **✔ Q-criterion definition and tensor split** (Eq. (1), Sec. 2.1, p.3)
   - **Claim:** Defines $Q = \frac{1}{2}(|\Omega|^2 - |S|^2)$ with Ω and S the antisymmetric/symmetric parts of ∇v.
   - **Checks:** notation consistency, dimensional/units, definition consistency
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** $|\cdot|$ denotes a consistent matrix norm (implicitly Frobenius)., ∇v is the velocity gradient tensor; transpose is well-defined.
   - **Notes:** Ω and S are defined consistently as antisymmetric/symmetric parts of ∇v. Q has units of $(\text{time})^{-2}$ if v has units length/time, consistent with interpreting $Q > 0$ as rotation-dominated regions. Minor naming ambiguity: Ω called "vorticity tensor" while $\omega = \nabla \times v$ is also defined.
2. **✔ Tracer equation of motion** (Eq. (2), Sec. 2.2, p.3)
   - **Claim:** Tracer positions satisfy $\frac{dx}{dt} = v(x(t), t)$.
   - **Checks:** dimensional/units, definition consistency
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** Tracers are passive and massless; no additional forces., v is sufficiently regular for RK4 integration conceptually.
   - **Notes:** Standard Lagrangian advection equation; units consistent.
3. **✔ Mean-Square Displacement definition** (Eq. (3), Sec. 2.3, p.3)
   - **Claim:** $\mathrm{MSD}(t) = \langle|\Delta x(t)|^2\rangle$ with $\Delta x(t) = x(t_0+t) - x(t_0)$.
   - **Checks:** notation consistency, dimensional/units
   - **Verdict:** PASS; confidence: high; impact: moderate
   - **Assumptions/inputs:** Ensemble average over tracers (and possibly t_0) is well-defined., Δx is a 3D vector displacement.
   - **Notes:** Definition is standard; however, under periodic boundaries Δx needs an explicit convention (unwrapped vs wrapped vs minimum-image) to interpret long-time behavior and saturation claims elsewhere.
4. **✔ Log-log slope transport exponent** (Sec. 2.3, p.4 (definition of $\alpha(t)$))
   - **Claim:** Defines $\alpha(t) = \frac{d(\log \mathrm{MSD})}{d(\log t)}$ and uses $\alpha = 2$ (ballistic), $\alpha = 1$ (diffusive).
   - **Checks:** definition consistency, sanity/limiting cases
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** $\mathrm{MSD}(t) > 0$ and differentiable/smooth enough for a local slope.
   - **Notes:** Consistent with $\mathrm{MSD} \propto t^{\alpha}$. Ballistic ($t^2$) and diffusive ($t^1$) interpretations match the definition.
5. **✔ Parallel MSD projection** (Eq. (4), Sec. 2.3, p.4)
   - **Claim:** Defines $\mathrm{MSD}{\parallel}(t)=\langle(\Delta x\cdot\hat{V} is the unit large-scale velocity direction.})^2\rangle where $\hat{V}_{\rm LS
   - **Checks:** algebra, units, definition consistency
   - **Verdict:** PASS; confidence: high; impact: moderate
   - **Assumptions/inputs:** $\hat{V}{\rm LS}isnormalizedanddefinedwhereverused(V\ne0$ or handled).

- **Notes:** Projection yields a scalar displacement component squared; units are length^2. Potential edge case $V_{\rm LS} = 0$ not discussed; would need handling in practice but is not an algebraic inconsistency.

6. ✔ **Perpendicular MSD projection** (Eq. (5), Sec. 2.3, p.4)
   - **Claim:** Defines $\mathrm{MSD} \perp(t)=\langle|\Delta x - (\Delta x\cdot\hat{V}|^2\rangle$.})\hat{V}_{\rm LS
   - **Checks:** algebra, units, definition consistency
   - **Verdict:** PASS; confidence: high; impact: moderate
   - **Assumptions/inputs:** $\hat{V}{\rm LS}isaunitvector., Same \hat{V}$ is used in both terms for each tracer/lag.
   - **Notes:** This is the squared norm of the component orthogonal to $\hat{V}_{\rm LS}$, computed via standard orthogonal projection removal.

7. ✔ **Orthogonal decomposition identity (implied)** (Implied by Eqs. (3)–(5), Sec. 2.3, p.4)
   - **Claim:** Implied that $|\Delta x|^2 = (\Delta x\cdot\hat{V}{\rm LS})^2 + |\Delta x - (\Delta x\cdot\hat{V}})\hat{V{\rm LS}|^2, so \mathrm{MSD} = \mathrm{MSD}\parallel + \mathrm{MSD}_\perp$.
   - **Checks:** algebra, consistency across definitions
   - **Verdict:** PASS; confidence: high; impact: moderate
   - **Assumptions/inputs:** $\hat{V}_{\rm LS}$ is a unit vector.
   - **Notes:** For any vector a and unit e: $a = (a \cdot e)e + (a - (a \cdot e)e)$ with orthogonal components. Taking squared norms gives the identity exactly.

8. ✔ **Anisotropy ratio definition** (Sec. 2.3, p.4)
   - **Claim:** Defines $\lambda(t)=\mathrm{MSD} \parallel(t)/\mathrm{MSD}\perp(t)$.
   - **Checks:** definition consistency, sanity/limiting cases
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** $\mathrm{MSD}_\perp(t) > 0$ in the regime of interest.
   - **Notes:** Dimensionless ratio; interpretation $\lambda < 1$ corresponds to greater perpendicular dispersion as stated.

9. ✔ **Isotropic expectation for λ (sanity check)** (Sec. 3.2 interpretation, p.6–p.7)
   - **Claim:** Late-time $\lambda \approx 0.52$ indicates perpendicular dispersion exceeds parallel dispersion; magnitude suggests about twice as effective transverse transport.
   - **Checks:** sanity/limiting cases, interpretation consistency
   - **Verdict:** PASS; confidence: medium; impact: minor
   - **Assumptions/inputs:** In 3D isotropy relative to an independent unit direction would give $\langle(\Delta x \cdot e)^2\rangle : \langle|\Delta x_\perp|^2\rangle = 1/3 : 2/3$
.
   - **Notes:** If Δx is isotropic and e is not systematically aligned, λ should be near $1/2$. Their reported stabilized $\lambda \approx 0.52$ is consistent with that benchmark; their "nearly twice" statement matches $1/0.52 \approx 1.9$.

10. ✖ **Periodic-domain geometric saturation value** (Sec. 3.1, p.5; Fig. 2 caption (p.6))
    - **Claim:** As tracers explore the periodic domain (side length L), the MSD approaches a theoretical saturation value $L^2/6$.
    - **Checks:** dimensional/units, consistency across definitions, sanity/limiting cases
    - **Verdict:** FAIL; confidence: medium; impact: critical
    - **Assumptions/inputs:** MSD is defined by Eq. (3) as $\langle|\Delta x|^2\rangle$., Domain is 3D periodic cube of side L.
    - **Notes:** With Eq. (3) using the full 3D squared distance, the saturation level depends on the displacement convention under periodicity. Common conventions (e.g., minimum-image per component in $[-L/2, L/2]$, or raw wrapped coordinate differences between two uniform points) do not yield total $\langle|\Delta x|^2\rangle = L^2/6$ in 3D. The paper does not define the convention used for Δx in a periodic domain, so the stated $L^2/6$ value is inconsistent with the stated MSD definition unless MSD is actually per-component or Δx is defined in a special way. This directly affects regime identification and interpretation at late times.

11. ✔ **Trapping-time definition from Q autocorrelation** (Sec. 2.4, p.4)
    - **Claim:** Defines τ_Q as the lag where the Lagrangian autocorrelation of Q decays to $1/e$.
    - **Checks:** definition consistency, units
    - **Verdict:** PASS; confidence: medium; impact: minor
    - **Assumptions/inputs:** Autocorrelation is normalized by its zero-lag value., $Q(t)$ along trajectories is stationary enough for this summary time scale.
    - **Notes:** Using the $1/e$ decay time is a consistent timescale definition provided the correlation is normalized. The normalization step is implied but not explicitly stated.

12. ✔ **Large-eddy turnover time formula and ratio** (Sec. 3.3, p.6–p.7)
    - **Claim:** Uses $T_e = L/v_{\rm rms}$ and compares τ_Q/T_e.
    - **Checks:** dimensional/units, algebra
    - **Verdict:** PASS; confidence: high; impact: minor
    - **Assumptions/inputs:** L is a length; $v_{\rm rms}$ a speed.
    - **Notes:** T_e has units of time; τ_Q/T_e is dimensionless. Algebraic manipulation shown is consistent.

13. ⚠ **Heavy-tail/Lévy-flight criterion vs Hill exponent sign** (Sec. 3.4, p.8)

   - **Claim:** States $\alpha_L \approx -7.1$ from Hill estimator and claims this confirms absence of Lévy-flight heavy tails ($0 < \alpha_L \leq 2$).
   - **Checks:** notation consistency, definition consistency
   - **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
   - **Assumptions/inputs:** α_L denotes a tail exponent in a power-law model., Same α_L symbol is used consistently across the criterion and estimate.
   - **Notes:** Without the explicit tail model and Hill-estimator definition, the negative exponent conflicts with the stated positive-range criterion. The qualitative conclusion "no heavy tails" may still be true, but the internal notation/sign convention is not verifiable from the paper as written.

14. ⚠ **FTLE discussion without definition** (Sec. 3.5, p.8–p.10)

   - **Claim:** Uses a long-time FTLE over the full trajectory; reports distributions centered near zero with slightly negative means.
   - **Checks:** definition consistency, sanity/limiting cases
   - **Verdict:** UNCERTAIN; confidence: low; impact: minor
   - **Assumptions/inputs:** FTLE computed as a log growth rate of infinitesimal or finite separations over a fixed time window.
   - **Notes:** No FTLE formula is provided, so it is impossible to audit sign conventions and whether the FTLE is the maximal exponent (typically nonnegative by definition) or a signed rate for a chosen direction/separation definition in a periodic domain.

Limitations

- Only the provided PDF text/images were used; several quantities central to claims (VACF-to-D relation, FTLE definition, Hill estimator definition) are referenced but not given as explicit equations, preventing full symbolic verification.
- No explicit statement is given for how displacement vectors are computed under periodic boundary conditions (unwrapping/minimum-image/wrapped), which is necessary to analytically validate late-time MSD saturation behavior.
- The audit does not assess numerical values, plots, fits, or simulation correctness; it checks only internal symbolic/analytic consistency.

Numerical results audit

This section audits **numerical/empirical** consistency: reported metrics, experimental design, baseline comparisons, statistical evidence, leakage risks, and reproducibility.

Eight text-based numeric consistency checks were performed (Pages 5–10). All checks passed within the stated tolerances, supporting internal arithmetic and interpretation consistency for the audited items (e.g., $L^2/6$ saturation, turnover-time computation, ratio-to-percent conversion, anisotropy interpretation, and tracer boundary-crossing count). Several other quantitative claims in the document could not be recomputed from text alone due to missing underlying datasets or figure-derived values.

Checked items

1. ✔ **C1_L2_over_6_saturation** (Page 5, Section 3.1 (also mentions in Figure 2 caption): "theoretical saturation value of $L^2/6 \approx 0.167$" with $L = 1$

   - **Claim:** As tracers explore the entire periodic domain of side length $L = 1$, their maximum possible squared displacement is limited, approaching the theoretical saturation value of $L^2/6 \approx 0.167$.
   - **Checks:** formula_substitution
   - **Verdict:** PASS
   - **Notes:** Computed $L^2/6 = 0.16666666666666666$; reported 0.167 matches as 3-decimal rounding.

2. ✔ **C2_turnover_time_from_L_vrms** (Page 6, Section 3.3: "$T_e = L/v_{\rm rms} \approx 1/0.38 \approx 2.6$")

   - **Claim:** Large-eddy turnover time T_e is computed as $T_e = L/v_{\rm rms} \approx 1/0.38 \approx 2.6$.
   - **Checks:** arithmetic_recomputation
   - **Verdict:** PASS
   - **Notes:** Computed $T_e = 2.6315789473684212$; difference consistent with rounding of $v_{\rm rms}$ and T_e.

3. ✔ **C3_tauQ_over_Te_ratio** (Page 6, Section 3.3: "$\tau_Q \approx 0.200 ... T_e \approx 2.6 ... \tau_Q/T_e \approx 0.077$")

   - **Claim:** Given $\tau_Q \approx 0.200$ and $T_e \approx 2.6$, the ratio τ_Q/T_e is reported as ≈ 0.077.
   - **Checks:** ratio_consistency
   - **Verdict:** PASS
   - **Notes:** Computed $0.2/2.6 = 0.07692307692307693$, consistent with 0.077 rounding.

4. ✔ **C4_ratio_to_percent_range_7_8** (Page 6-7, Section 3.3 and Figure 4 caption: "$\tau_Q/T_e \approx 0.077$ ... only about 7–8%")

   - **Claim:** The ratio $\tau_Q/T_e \approx 0.077$ is described as "only about 7–8% of a large-eddy turnover time."
   - **Checks:** unitless_to_percent_conversion
   - **Verdict:** PASS
   - **Notes:** Computed $100 \times 0.077 = 7.7\%$, which lies within the stated 7–8% range.

5. ✔ **C5__anisotropy__mean__vs__uncertainty__significance** (Page 6, Section 3.2: "$\lambda \approx 0.52 \pm 0.045$" and claim "stabilizes below unity")
    - **Claim:** Mean anisotropy ratio is $\lambda \approx 0.52 \pm 0.045$ and is claimed to stabilize below unity.
    - **Checks:** inequality__with__uncertainty
    - **Verdict:** PASS
    - **Notes:** Conservative bound check: $0.52 + 0.045 = 0.565 < 1.0$, consistent with "below unity."
6. ✔ **C6__anisotropy__implied__perp__over__parallel** (Page 10, Conclusions: "$\lambda \approx 0.52$ ... transverse transport is nearly twice as effective as parallel transport.")
    - **Claim:** With anisotropy ratio $\lambda = \mathrm{MSD} *\parallel/\mathrm{MSD}*\perp \approx 0.52$, the text claims transverse transport is nearly twice as effective as parallel transport.
    - **Checks:** inverse__ratio__interpretation
    - **Verdict:** PASS
    - **Notes:** Implied $\mathrm{MSD} *\perp/\mathrm{MSD}*\parallel = 1/0.52 = 1.923...$, consistent with "nearly twice."
7. ✔ **C7__tracers__crossed__boundary__count** (Page 5, Section 3.1: "By the end of the simulation, 82.9% of tracers have crossed a periodic boundary" with $N = 8,000$)
    - **Claim:** 82.9% of $8,000$ tracers have crossed a periodic boundary.
    - **Checks:** percent__to__count
    - **Verdict:** PASS
    - **Notes:** Computed crossed count = $8000 \times 0.829 = 6632$ exactly (an integer), consistent with the stated percentage.
8. ✔ **C8__FTLE__means__difference__vs__reported** (Page 9, Section 3.5 and Figure 6 caption: means -0.008 ± 0.013 (Trapped) vs -0.007 ± 0.013 (Free))
    - **Claim:** Mean FTLE values are -0.008 ± 0.013 for Trapped and -0.007 ± 0.013 for Free; described as statistically indistinguishable.
    - **Checks:** difference__vs__uncertainty
    - **Verdict:** PASS
    - **Notes:** Heuristic consistency: $|-0.008 - (-0.007)| = 0.001$, which is smaller than 0.013; not a formal significance test without sample size.

Limitations

- Audit is restricted to the provided PDF text; no external datasets, code, or supplementary material were used.
- No numeric values were extracted from plots/figures by pixel reading; only explicit numbers in the text/captions were used.
- Several statistical claims (KS p-values, kurtosis, Hill estimator exponent, Pearson r) cannot be recomputed without the underlying tracer data or distributions.