Skepthical review: *Probing the Asymptotic Link Between Eulerian Roughness and Fractional Lagrangian Diffusion in Turbulence*

Summary

The manuscript probes the proposed asymptotic RG-style mapping between Eulerian velocity-field roughness ξ and a Lagrangian fractional-transport (Lévy) index α, stated throughout as $\alpha = 2/\xi$ (Abstract; Sec. 1; Sec. 2.4; Sec. 4). The authors estimate Eulerian scaling exponents from structure functions across several model systems (1D multifractal cascades; Lorenz–96; 3D Kraichnan-type synthetic velocity fields; and a 1D synthetic Kolmogorov-like field), then analyze Lagrangian transport using MSD/Hurst exponents, displacement PDFs, tail-index tools (Hill-type plots), and a sliding-window effective $\alpha(\tau)$ inferred from characteristic functions (Sec. 2.1–2.4; Sec. 3). The main reported outcome is that Eulerian scalings look broadly consistent with expectations (Sec. 3.1–3.2), but the predicted Lagrangian Lévy regime is not convincingly observed in accessible 3D Kraichnan simulations (effective $\alpha(\tau)$ remains near 2; PDFs are close to Gaussian; Sec. 3.3–3.4), while in 1D the mapping fails strongly with pronounced subdiffusion attributed to topological trapping (Sec. 3.5). The question—whether and when such asymptotic mappings are observable in finite, discrete numerical settings—is timely and important, and the "empirical RG-flow" visualization via $\alpha(\tau)$ is a potentially useful diagnostic. However, the paper currently leaves key theoretical assumptions and even the target observable (single-particle displacement vs alternatives such as pair dispersion) insufficiently specified, mixes diagnostics that are not mutually consistent for ideal $\alpha < 2$ stable laws (MSD vs Lévy flights), and under-documents core simulation and inference details (Sec. 2; Fig. 2–8). Several central interpretations—pre-asymptotic trapping in 3D and a "fundamental breakdown" in 1D—therefore read as plausible but not yet quantitatively established. Strengthening the theory/definitions section, making the inference and numerical protocols reproducible, adding minimal parameter/robustness studies and uncertainty quantification, and reframing conclusions conditionally where assumptions are violated would substantially improve the manuscript's evidentiary chain and impact.

Strengths

- Targets a clear "observability" problem: whether an asymptotic $\alpha = 2/\xi$ mapping can actually be seen in finite-resolution/time simulations (Introduction; Sec. 4).
- Uses a multi-model hierarchy (multifractal cascades, 3D Kraichnan, Lorenz–96, 1D synthetic turbulence) that, in principle, can disentangle intermittency, temporal decorrelation, dimensionality, and deterministic vs stochastic dynamics (Sec. 2.1; Sec. 3.1–3.5).
- Eulerian analysis via structure functions is largely standard and provides a coherent baseline across models (Sec. 2.2; Sec. 3.1).

- The sliding-window characteristic-function approach to obtain an effective $\alpha(\tau)$ is a promising way to visualize crossovers and "RG-flow-like" behavior in transport exponents (Sec. 2.4; Sec. 3.4).
- The negative/nuanced results (near-Gaussian 3D transport despite rough Eulerian fields; strong 1D trapping) are potentially valuable to the community if backed by tighter theory and quantitative diagnostics (Sec. 3.3–3.5; Sec. 4).

Major issues

1. **The theoretical status, assumptions, and even the intended observable behind the central mapping $\alpha = 2/\xi$ are not stated precisely, making it hard to interpret deviations as (i) genuine counterexamples, (ii) pre-asymptotic limitations, or (iii) regime mismatch (Abstract; Sec. 1; Sec. 2.4; Sec. 3.3–3.5; Sec. 4).** In particular: (a) the definition of ξ is ambiguous due to mixed conventions (Sec. 2.1–2.2; Eq. (2); the statement $\xi = 1 + \zeta_2$ and the Kraichnan spectrum parameterization $E(k) \propto k^{-(1+\zeta_2)}$); (b) it is not clearly justified that single-particle displacement in the specific 3D Kraichnan setting should converge to a space-fractional α-stable law (many classic Kraichnan anomalous results concern pair dispersion/scalars and depend on compressibility/dimension); and (c) the paper does not explicitly list the hypotheses under which the RG fixed point is expected (e.g., dimensionality, incompressibility/compressibility, isotropy, white-in-time vs finite correlation, absence of strong trapping, intermittency relevance).

   Recommendation: Add a dedicated theory/conventions subsection early (e.g., new Sec. 1.1 or Sec. 2.0) that: (i) defines ξ unambiguously in the paper's convention (e.g., $\langle|\delta u(r)|^2\rangle \sim r^\xi$ or via $E(k)$), and explicitly derives/quotes the relation between the spectrum exponent and structure-function exponent in the dimensions used (disambiguate symbols, e.g., $\zeta_{2,\mathrm{SF}}$ vs a spectral slope parameter); (ii) states what Lagrangian quantity is predicted to be α-stable (single-particle displacement component? radial displacement? increments?) and under what model class (Gaussian, homogeneous, isotropic, incompressible?, white-in-time); (iii) clarifies whether the mapping pertains to single-particle or pair dispersion and why the chosen observable is the correct test in your settings; (iv) lists assumptions/known limitations (dimension, compressibility, trapping/topology) and, for each model in Sec. 3.1–3.5, explicitly states which assumptions hold/violate; (v) adds primary references where $\alpha = 2/\xi$ (and any accompanying relation such as $H = \xi/2$) is derived/discussed for turbulent transport, distinguishing rigorous vs heuristic arguments. Then adjust the language in Sec. 4 from unconditional "universal fixed point" to conditional statements tied to these hypotheses.

2. **There is an internal mismatch between the Lévy/space-fractional picture ($\alpha < 2$) and the heavy reliance on MSD/Hurst exponents as a primary validation target (Sec. 2.3; Sec. 3.3; the MSD-based benchmark $H = \xi/2$; also**

noted around p.6–7). For ideal symmetric α-stable Lévy flights with $\alpha < 2$, $\langle \Delta x^2 \rangle$ diverges, so MSD-based scaling is not a mathematically consistent asymptotic diagnostic unless a specific regularization is part of the model (finite domain, truncation/tempering, finite-resolution cutoffs). As written, MSD is used both to assess "diffusive" behavior and to compare against a fractional model whose second moment need not exist, which can bias conclusions (e.g., toward $\alpha \approx 2$ under finite-variance/finite-sample regimes).

Recommendation: Make the modeling choice explicit and consistent: either (a) pivot the primary Lagrangian scaling tests to observables compatible with $\alpha < 2$, e.g., fractional moments $\langle |\Delta x|^q \rangle$ for $q < \alpha$, inter-quantile scaling, or scaling of the typical displacement (median/quantiles); or (b) explicitly define a regularized/tempered/truncated stable model induced by finite inertial range, periodic domain, or numerical cutoff, and derive what MSD scaling should be expected under that regularization (including how it depends on resolution and time). Then rewrite Sec. 3.3–3.5 so that "agreement/disagreement" is judged using observables whose asymptotics match the stated model class.

3. **The Lagrangian inference pipeline for α and especially $\alpha(\tau)$ is under-specified and lacks calibration/robustness checks, so central conclusions ("$\alpha(\tau)$ pinned near 2" in 3D; extremely small α in 1D) cannot be independently assessed (Sec. 2.3–2.4; Sec. 3.3–3.5; Fig. 6–8).** Key missing details include: number of tracers/realizations and trajectory lengths; sampling interval; how $\phi(k, \tau) = \langle e^{ik\Delta x(\tau)} \rangle$ is computed (k-grid, averaging, overlapping windows); the fit strategy for $\exp(-D(\tau)|k|^{\alpha(\tau)})$ (joint fit vs sequential, weights, goodness-of-fit criteria, treatment of large-k noise/aliasing); and uncertainty quantification. Moreover, without careful k-window choice, finite-variance/truncated tails can make the small-k behavior look quadratic, biasing α toward 2 even for non-Gaussian laws. The use of Hill plots/tail estimators also needs justification for symmetric stable/tempered-stable settings.

   Recommendation: Expand Sec. 2.3–2.4 into a reproducible protocol: (i) per model, report tracer count, realizations, total time, Δt, and the ratio of analyzed lags to correlation time τ_c; (ii) specify whether Δx is a 1D component or 3D vector magnitude, and update Eq. (4) accordingly (vector $\mathbf{k}$ vs scalar k); (iii) define the k-grid and the exact fit window $[k_{\min}, k_{\max}]$ used for each τ, with a rule for excluding noisy/aliased points; (iv) state the objective function and whether $D(\tau)$ and $\alpha(\tau)$ are jointly fit, plus goodness-of-fit diagnostics; (v) add bootstrap/subsampling confidence intervals for α, $\alpha(\tau)$, and any reported plateaus; (vi) validate the estimator on synthetic data with known α (and with finite-sample sizes comparable to your simulations) to demonstrate identifiability and bias. For tail indices, either justify Hill under your assumptions or replace/augment with stable-law-oriented estimators (e.g., characteristic-function regression methods, stable MLE with diagnostics) and report estimator stability to threshold choices.

4. **The key negative result in the 3D Kraichnan model—near-Gaussian transport and lack of approach to the predicted $\alpha = 2/\xi$ regime—is attributed to finite spectral resolution and finite time ("pre-asymptotic trapping"), but the numerical setup is not fully specified and the attribution is not supported by a systematic parameter/finite-size study (Sec. 2.1–2.3; Sec. 3.3–3.4).** Crucial missing details include: box size and boundary conditions (periodicity and wrap-around effects on displacement statistics), wavenumber range $[k_{\min}, k_{\max}]$, number of Fourier modes N_k in each direction and incompressibility projection, how white-in-time is implemented, forcing/dissipation or spectral shaping, tracer integrator and time step, total integration time and number of statistically independent samples. With a single (or very limited) resolution/time horizon, it remains unclear whether the observed Gaussianity reflects genuine inapplicability of the mapping to the chosen observable, or merely insufficient scale separation.

   Recommendation: In Sec. 2.1–2.3 add a complete Kraichnan-implementation description and summarize it in a parameter table (also including tracer counts, Δt, T, k-range, boundary conditions). Then, in Sec. 3.3–3.4 provide at least a minimal finite-size/time study: vary N_k and/or inertial-range width $k_{\max}/k_{\min}$, and/or total time T, and show how (a) PDF non-Gaussianity (e.g., kurtosis/flatness), (b) $\alpha(\tau)$ plateaus/crossovers (with CI), and (c) any alternative consistent scaling observable (see Major Issue 2) change. If computational constraints prevent this, explicitly state the limitation and rephrase "pre-asymptotic trapping explains" into "consistent with a finite-size/time explanation; not demonstrated here."

5. **The 1D synthetic turbulence section argues a "fundamental breakdown" of $\alpha = 2/\xi$ due to "topological trapping," but the mechanism is not quantitatively diagnosed and the appropriate effective model may not be space-fractional Lévy flight at all (Sec. 3.5; Sec. 4).** Strong subdiffusion (e.g., very small H) is often associated with waiting-time mechanisms (CTRW/subordination; time-fractional diffusion), whereas the paper fits space-fractional α-stable forms. In addition, it is not clarified whether the RG mapping was ever expected to hold in 1D (many results are for $d \geq 2$) and how specific modeling choices (boundary conditions, temporal correlations, stagnation-point statistics) control trapping.

   Recommendation: First, in Sec. 2.1 precisely define the 1D synthetic velocity field (construction, spectrum, intermittency parameters, temporal correlation, domain/boundaries). Second, in Sec. 3.5 add quantitative trapping diagnostics: e.g., residence-time distributions in low-|u| regions, waiting-time statistics near stagnation points, distribution/density of velocity zeros, or displacement autocorrelation signatures; include representative trajectory visualizations tied to those statistics. Third, discuss model class: test whether a time-fractional/subordinated model better captures the observed subdiffusion (e.g., fit waiting-time tails and relate to a time-fractional order), and clearly separate "outside assumptions of $\alpha = 2/\xi$" from "counterexample within assumptions." Finally, soften "fundamental breakdown" unless you can cite theory predicting the mapping for your 1D setting.

6. **The Lorenz–96 component is currently underdeveloped relative to how it is motivated (Sec. 2.1–2.3 vs Sec. 3.1–3.2).** It is presented as a deterministic-chaotic contrast, but the Results show mainly Eulerian structure-function exponents and a decorrelation time, without the corresponding Lagrangian transport analysis (MSD alternatives, PDFs, α, $\alpha(\tau)$). This leaves an incomplete "bridge" in the narrative and weakens the manuscript's claim to test robustness across stochastic vs deterministic dynamics.

   Recommendation: Either (i) add a Lorenz–96 Lagrangian subsection (e.g., Sec. 3.6) that mirrors the Kraichnan diagnostics (using observables consistent with your chosen fractional model), including $\alpha(\tau)$ with uncertainty, and discuss how deterministic chaos affects (or fails to affect) approach to the predicted mapping; or (ii) explicitly reposition Lorenz–96 as an Eulerian-only benchmark and remove/temper any claims that it informs Lagrangian fractional transport. In both cases, document the Lorenz–96 configuration in Sec. 2.1 (N, forcing F, integration method/Δt, simulation length, and how the discrete index is treated as a spatial coordinate).

Minor issues

1. Uncertainty quantification is largely absent across key reported outputs (ζ_p, ξ, H, PDF fits, Hill/CF estimates, $\alpha(\tau)$) and most figures lack explicit benchmarks for theoretical predictions (Fig. 2–8; Sec. 3). This makes it difficult to tell whether deviations (e.g., H near 0.5) are statistically meaningful.

   Recommendation: Add confidence intervals/error bands via bootstrap across tracers/realizations (and, for Eulerian statistics, across snapshots/segments). In captions, state sample sizes and fit windows. Overlay reference slopes/lines for predicted exponents (e.g., predicted $\alpha = 2/\xi$ and any expected scaling regimes) directly on plots.

2. Model/experiment parameterization is scattered, limiting reproducibility and making finite-size/time discussions hard to follow (Sec. 2.1–2.3; Sec. 3.1–3.5).

   Recommendation: Provide a single parameter table for each model: domain size, resolution/modes, boundary conditions, forcing/spectrum parameters, correlation time τ_c, Δt, total time T, number of realizations, number of tracers, and seeding strategy; also list measured ζ_2/ξ and predicted vs inferred α with CI.

3. Eulerian exponent extraction (inertial-range selection and uncertainty on ζ_p and ξ) is not described in enough detail (Sec. 2.2; Sec. 3.1), which matters because ξ is the input to the central mapping.

Recommendation: Describe the inertial-range identification criterion (local-slope plateau, fixed fraction of domain, R^2 thresholds), number of points used, and provide uncertainty estimates (bootstrap or variability across realizations). Where possible, show compensated plots or local slopes to justify fit ranges.

4. Temporal decorrelation is not reported systematically across models, yet it conditions both the "white-in-time" assumption and the selection of lag windows for transport fits (Sec. 2.3; Sec. 3.2).

   Recommendation: For each model report $R_v(\tau)$ and a consistent definition of τ_c; state the analyzed lag range in units of τ_c (e.g., $\tau/\tau_c \in [\cdot, \cdot]$) and briefly discuss possible bias from residual correlations.

5. Notation and dimensional consistency issues persist between 1D and 3D contexts (Sec. 2.2–2.4): scalar vs vector displacement in Eq. (4), and how ξ is defined/estimated in 3D vs from 1D "cuts."

   Recommendation: Explicitly define the 3D ξ used (e.g., longitudinal increments or spectrum in d=3) and state whether you analyze a displacement component or isotropic magnitude. Update the characteristic function definition accordingly and keep notation consistent (e.g., $|\mathbf{k}|$, $|\mathbf{\Delta x}|$ if used).

6. Some interpretations appear overstated without error bars (e.g., labeling values extremely close to 0.5 as clearly subdiffusive) (Sec. 3.3; Sec. 3.5).

   Recommendation: Report uncertainties for H and avoid categorical labels unless deviations are significant relative to CI; otherwise describe as "consistent with diffusion within uncertainty" vs "clear subdiffusion."

7. Figure readability/accessibility: several panels are dense, legends overlap data, axes lack units/normalization, and color/line distinctions may not be robust for print or color-vision deficiencies (Fig. 2–8).

   Recommendation: Increase font/panel sizes, use colorblind-safe palettes with line-style redundancy, move legends away from data, label axes with units/normalizations, and export as vector/high-DPI. Where tails or plateaus matter, add insets/zoomed panels and mark fit windows.

8. The reference list does not consistently anchor the key claims (RG mapping, Kraichnan transport, anomalous diffusion), and some citations appear misaligned with the statements they support (Sec. 1–2; References).

   Recommendation: Add/replace with foundational references on fractional kinetics/anomalous diffusion and Kraichnan-model transport relevant to your specific observable (single-particle vs pair dispersion) and to α-stable vs time-fractional mechanisms; ensure each major theoretical claim is supported by an appropriate citation.

9. Nonstandard affiliation wording ("Anthropic, Gemini & OpenAI servers") may be inappropriate for an academic submission and distracts from the content (front matter).

   Recommendation: Replace with standard institutional affiliations or remove/relocate to acknowledgments in line with the target journal's policies.

Very minor issues

1. Typos/formatting artifacts and inconsistent section/equation referencing (split words, stray symbols in captions, inconsistent heading styles, mixed use of t vs τ) reduce polish and occasionally clarity (Sec. 1; Sec. 3.3–3.5; captions; References).

   Recommendation: Proofread to remove line-break artifacts, standardize headings, ensure consistent notation (use τ for lags and t for absolute time), and reference key equations (e.g., Eq. (4) for ϕ) by number when used later.

2. Diffusivity notation is inconsistent between the fractional PDE coefficient (e.g., D_α) and the fitted effective $D(\tau)$ (Sec. 2.4).

   Recommendation: Add a short bridging definition explaining whether $D(\tau)$ is purely empirical, or how it relates to a constant-coefficient fractional model under your assumptions/regularization.

Key statements and references

- **✖ The theoretical Renormalization Group prediction for turbulent transport states that, in the asymptotic limit of long times and large scales, the Lagrangian Lévy exponent α is determined by the Eulerian spectral roughness ξ of the velocity field via the fixed-point relation $\alpha = 2/\xi$, where ξ is defined from the second-order structure function scaling exponent ζ_2 through $\xi = 1 + \zeta_2$, so that increased intermittency (and thus larger ξ) should correspond to smaller α and more anomalous, heavy-tailed transport.**
- *Reference(s):* [10], [11]
- *Justification:* The paper [11] analyzes passive-scalar decay in the Kraichnan model, defining α as a decay/scaling exponent of self-similar solutions and ζ (with $\gamma = 2 - \zeta$) as the Eulerian velocity roughness parameter. It provides Kummer-function solutions and realizability bounds (e.g., $\alpha \le d + \gamma$), and discusses structure-function behavior, but it does not present a renormalization-group fixed-point relation $\alpha = 2/\xi$, does not define $\xi = 1 + \zeta_2$, nor link a Lagrangian Lévy exponent to Eulerian intermittency. Hence the stated RG prediction is not supported by [11].

- **✖ In the multifractal energy cascade model used for the Eulerian analysis, intermittency is controlled by the parameter μ in a random multiplicative cascade construction, and for the non-intermittent Kolmogorov-like case (**

$\mu = 0.0$) the measured structure function exponents follow $\zeta_p = p/3$ with $\zeta_2 = 2/3$, yielding a spectral roughness $\xi = 5/3 \approx 1.667$ in agreement with Kolmogorov theory, while increasing μ to 0.15 and 0.28 produces non-linear, concave ζ_p consistent with multifractal theory and roughness values $\xi = 1.683$ and $\xi = 1.697$, respectively.

- *Reference(s):* [4], [5]
- *Justification:* [4] models intermittency with parameters K and p in a binomial multiplicative cascade and derives expressions for D_q, ζ_p, and a steeper energy spectrum when $0 < K < 1$. It does not introduce or use an intermittency parameter μ, does not present measured ζ_p curves, nor report spectral roughness values $\xi = 1.683$ or 1.697. The specific claims about $\mu = 0.0$, 0.15, 0.28, ζ_p concavity, and ξ values are not provided in [4].

- ✖ **In the three-dimensional Kraichnan model, whose Gaussian, homogeneous, isotropic velocity field is constructed to be delta-correlated in time and to have a power-law energy spectrum $E(k) \propto k^{-(1+\zeta_2)}$ so that the spectral roughness $\xi = 1 + \zeta_2$ can be prescribed as a control parameter, the theoretical expectation is superdiffusive transport with Hurst exponent $H = \xi/2$ and Lévy index $\alpha = 2/\xi$, yet numerical tracer simulations at finite spectral resolution ($N_k = 128$) and finite duration show Mean Squared Displacement scaling with $H \approx 0.5$ and displacement PDFs with Gaussian-dominated cores, while Hill-plot estimates of α remain pinned near the Gaussian limit $\alpha \approx 2$ and fail to converge to $2/\xi$, indicating pre-asymptotic trapping where long-range spatial correlations are insufficiently sampled to realize the predicted fractional regime within accessible times. **

- *Reference(s):* [6], [11], [12]
- *Justification:* While [11] describes the Kraichnan model (Gaussian, homogeneous, isotropic, delta-correlated in time with a spatial roughness exponent) and its spectral scaling, it does not discuss single-particle superdiffusion, Hurst exponents $H = \xi/2$, or Lévy indices $\alpha = 2/\xi$, nor any tracer MSD/PNG results. [12] concerns ML-generated Lagrangian data for Navier–Stokes turbulence (not the Kraichnan model) and does not report MSD scaling, Lévy α, Hill plots, or simulations at $N_k = 128$. Thus the claimed theoretical expectations and numerical findings are not supported by [11], [12].

Mathematical consistency audit

This section audits **symbolic/analytic** mathematical consistency (algebra, derivations, dimensional/unit checks, definition consistency).

Maths relevance: light

The paper contains a small set of explicit equations (structure functions, roughness definition, autocorrelation, characteristic function) and several central scaling/mapping claims ($\alpha = 2/\xi$, and a claimed MSD/Hurst prediction $H = \xi/2$). Most theoretical links are asserted rather than derived. The main internal mathematical concern is the simultaneous use of an α-stable ($\alpha < 2$) Lévy-flight framework and MSD-based scaling predictions, which are not compatible without explicit regularization.

Checked items

1. ✔ **Fractional diffusion equation form** (Sec. 1, p.2 (unnumbered equation: $\partial_t P = -D_\alpha(-\Delta)^{\alpha/2} P$))
    - **Claim:** Tracer PDF evolves under a fractional diffusion equation with Lévy exponent α, recovering normal diffusion at $\alpha = 2$.
    - **Checks:** operator/sign sanity, dimensional/units sanity, notation consistency
    - **Verdict:** PASS; confidence: medium; impact: moderate
    - **Assumptions/inputs:** $(-\Delta)^{\alpha/2}$ is the (positive) fractional Laplacian operator, $D_\alpha > 0$
    - **Notes:** The sign convention is consistent with a generator written as $-D_\alpha(-\Delta)^{\alpha/2}$. Units would require $[D_\alpha] = L^\alpha/T$, but units are not discussed; no contradiction appears internally.

2. ✔ **Structure function definition** (Eq. (1), Sec. 2.2, p.3)
    - **Claim:** $S_p(r) = \langle |u(x + r) - u(x)|^p \rangle$ defines the p-th order structure function.
    - **Checks:** definition consistency
    - **Verdict:** PASS; confidence: high; impact: minor
    - **Assumptions/inputs:** $u(x)$ is a 1D velocity field snapshot/realization, $\langle\cdot\rangle$ is a spatial average
    - **Notes:** Standard definition; later references to ζ_p rely on this.

3. ✔ **Structure-function scaling ansatz** (Sec. 2.2, p.3 ($S_p(r) \propto r^{\zeta_p}$))
    - **Claim:** For statistically self-similar fields, $S_p(r)$ scales as a power law in r with exponent ζ_p.
    - **Checks:** definition consistency
    - **Verdict:** PASS; confidence: high; impact: minor
    - **Assumptions/inputs:** Existence of an inertial range with scaling, ζ_p defined by log-log slope
    - **Notes:** Used consistently for extracting ζ_p.

4. ✔ **Roughness definition from ζ_2** (Eq. (2), Sec. 2.2, p.4)
    - **Claim:** Spectral roughness is defined as $\xi = 1 + \zeta_2$.
    - **Checks:** symbol/definition consistency

- **Verdict:** PASS; confidence: medium; impact: critical
- **Assumptions/inputs:** ζ_2 is the second-order structure-function exponent
- **Notes:** As a definition it is internally consistent, but its connection to the spectrum $E(k)$ and to the RG mapping is not derived in the paper.

5. ⚠ **Kraichnan spectrum parameterization vs** ξ (Sec. 2.1, p.3)

   - **Claim:** Kraichnan energy spectrum $E(k) \propto k^{-(1+\zeta_2)}$ and the spectral roughness is $\xi = 1 + \zeta_2$.
   - **Checks:** symbol consistency, missing-derivation check
   - **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
   - **Assumptions/inputs:** Same ζ_2 appears in both spectrum exponent and in Eq. (2), The definition of ξ is consistent across 3D and 1D
   - **Notes:** The paper does not explicitly connect the ζ_2 in the spectral exponent to the ζ_2 defined from structure functions, nor does it define ξ for the 3D setting; symbol reuse makes this ambiguous.

6. ✔ **Velocity autocorrelation definition** (Eq. (3), Sec. 2.3, p.4)

   - **Claim:** $R_v(\tau) = \langle v(t) \cdot v(t+\tau) \rangle$ defines the Lagrangian velocity autocorrelation.
   - **Checks:** definition consistency
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** $v(t)$ is tracer velocity, Ensemble average over tracers/times
   - **Notes:** Internally consistent; later discussion of correlation time refers to this quantity.

7. ✔ **MSD scaling and Hurst exponent definition** (Sec. 2.3, p.4 ($\langle \Delta x^2(\tau) \rangle \propto \tau^{2H}$))

   - **Claim:** Transport regime is characterized by MSD scaling exponent $2H$.
   - **Checks:** definition consistency
   - **Verdict:** PASS; confidence: high; impact: moderate
   - **Assumptions/inputs:** Second moment exists/finite over the dataset, Scaling range in τ exists
   - **Notes:** Mathematically fine as an empirical definition, but conflicts with the ideal α-stable model when $\alpha < 2$ unless regularized (see separate item).

8. ⚠ **Central mapping** $\alpha = 2/\xi$ (Abstract; Sec. 1 (p.2); reiterated Sec. 3.1–4 (p.5–12))

   - **Claim:** Asymptotically, Lévy exponent α is determined by Eulerian roughness via $\alpha = 2/\xi$.
   - **Checks:** missing-derivation check, definition consistency across sections
   - **Verdict:** UNCERTAIN; confidence: low; impact: critical

- **Assumptions/inputs:** RG fixed point exists and is accessible, ξ is the correct roughness parameter entering the RG theory, Dimensionality/topology do not modify the mapping (later disputed for 1D)
- **Notes:** No derivation or formal statement of assumptions is provided in the paper text, so the mapping cannot be audited for correctness from within the PDF. Algebraic uses of the formula are consistent (e.g., $\xi = 5/3 \to \alpha = 1.2$).

9. ✖ **Claimed MSD prediction $H = \xi/2$** (Sec. 3.3, p.6 (text: "theory predicts ... MSD ... with $H = \xi/2$"))

   - **Claim:** The theory predicts superdiffusive MSD scaling with $H = \xi/2$, alongside $\alpha = 2/\xi$.
   - **Checks:** internal model-observable consistency, missing-derivation check
   - **Verdict:** FAIL; confidence: high; impact: critical
   - **Assumptions/inputs:** Underlying model is Lévy-stable/fractional diffusion with $\alpha = 2/\xi$, MSD is a valid observable for the predicted regime
   - **Notes:** Within the paper's own stated modeling choice (fractional diffusion/Lévy-stable with $\alpha < 2$ for $\xi > 1$), the second moment is not finite in the ideal asymptotic model, so an MSD scaling exponent H is not well-defined without explicit truncation/tempering or confinement. The paper presents $H = \xi/2$ as a theoretical MSD prediction but does not specify such a regularization.

10. ✔ **Characteristic function definition** (Eq. (4), Sec. 2.4, p.4)

    - **Claim:** $\phi(k, \tau) = \langle e^{ik\Delta x(\tau)} \rangle$ defines the empirical characteristic function of displacements.
    - **Checks:** definition consistency
    - **Verdict:** PASS; confidence: high; impact: minor
    - **Assumptions/inputs:** k is scalar or Δx is 1D-projected displacement
    - **Notes:** Consistent as written; dimensional/vector generalization is not specified for 3D.

11. ✔ **α-stable characteristic function ansatz used for fitting** (Sec. 2.4, p.4–5 ($\phi_{\alpha,D}(k, \tau) = \exp(-D(\tau)|k|^{\alpha(\tau)})$))

    - **Claim:** Displacement increments at lag τ are fit by a symmetric α-stable characteristic function with effective parameters $\alpha(\tau)$, $D(\tau)$.
    - **Checks:** algebraic/log-slope implication, notation consistency
    - **Verdict:** PASS; confidence: medium; impact: moderate
    - **Assumptions/inputs:** Symmetric (zero-skew) stable law, Any τ-scaling absorbed into $D(\tau)$

- **Notes:** Given the stated ansatz, $-\log\phi = D(\tau)|k|^{\alpha(\tau)}$ so log-log slope in $|k|$ yields $\alpha(\tau)$, consistent with Sec. 3.4's description. The relationship between D_α in the PDE and $D(\tau)$ is not specified.

12. ✔ **Inference that parallel log-log lines imply constant $\alpha(\tau)$** (Sec. 3.4, p.8–10 (discussion around Figs. 6–7))

    - **Claim:** If $-\log\phi(k,\tau)$ vs $|k|$ has parallel slopes for different τ, then $\alpha(\tau)$ is constant across τ.
    - **Checks:** algebraic implication
    - **Verdict:** PASS; confidence: medium; impact: minor
    - **Assumptions/inputs:** Each curve is in a k-range where the power-law ansatz applies, Slope corresponds to $\alpha(\tau)$ per the fit model
    - **Notes:** Follows directly from the stated fitting form; auditing the plotted slopes numerically is out of scope.

Limitations

- Only the PDF text provided was used; no supplementary material or full-resolution figure data were available for analytic cross-checks.
- Key theoretical relationships ($\alpha = 2/\xi$ and $H = \xi/2$) are asserted without derivation in the paper; where derivations are missing, items are marked UNCERTAIN or (if internally contradictory) FAIL.
- Model-specific definitions of ξ across 1D and 3D contexts are not formalized in the paper text, limiting the ability to audit dimensional/definition consistency across sections.

Numerical results audit

This section audits **numerical/empirical** consistency: reported metrics, experimental design, baseline comparisons, statistical evidence, leakage risks, and reproducibility.

Performed 8 internal-consistency recomputations from explicitly stated numbers (e.g., $\xi = 1 + \zeta_2$, $H = \xi/2$, $\alpha = 2/\xi$, τ_c/T ratio, and a time-lag range sanity check). All checks passed within the provided tolerances; several comparisons involved rounded decimals (e.g., 1.667 vs $5/3$).

Checked items

1. ✔ **C1** (Page 5, Sec. 3.1 (Kolmogorov-like cascade, $\mu = 0.0$))

    - **Claim:** For $\zeta_p = p/3$, they state $\zeta_2 = 2/3$ and $\xi = 5/3 \approx 1.667$.
    - **Checks:** algebraic_recompute
    - **Verdict:** PASS
    - **Notes:** Computed $\xi = 1 + \zeta_2 = 1.6666666666999999$; consistent with $\xi = 5/3$ and rounded 1.667 within tolerance.

2. ✔ **C2** (Page 3, Sec. 2.2 and Page 5, Sec. 3.1 (Lorenz-96 roughness))

- **Claim:** They state $\zeta_2 = 0.029$ and $\xi = 1.029$ for Lorenz-96, consistent with $\xi = 1 + \zeta_2$.
- **Checks:** algebraic_recompute
- **Verdict:** PASS
- **Notes:** Exact match for $\xi = 1 + \zeta_2$.

3. ✔ **C3** (Page 5, Sec. 3.1 (Intermittency cases))

   - **Claim:** They report $\xi = 1.683$ $(\mu = 0.15)$ and $\xi = 1.697$ $(\mu = 0.28)$; check implied ζ_2 values via $\zeta_2 = \xi - 1$.
   - **Checks:** derived_value_check
   - **Verdict:** PASS
   - **Notes:** Back-computed $\zeta_{2,\text{mild}} = 0.683$ and $\zeta_{2,\text{realistic}} = 0.6970000000000001$; ordering check $\zeta_{2,\text{realistic}} > \zeta_{2,\text{mild}} > 2/3$ passed.

4. ✔ **C4** (Page 6, Sec. 3.2 (Lorenz-96 correlation time vs duration))

   - **Claim:** They state Lorenz-96 has $\tau_c = 0.23$ relative to total simulation duration $T = 500$; check scale separation ratio.
   - **Checks:** ratio_recompute
   - **Verdict:** PASS
   - **Notes:** Computed $\tau_c/T = 0.00046$; matches and also satisfies the suggested ratio $< 1\text{e}{-3}$ condition.

5. ✔ **C5** (Page 6, Sec. 3.3 (Example superdiffusion prediction))

   - **Claim:** They give example $H = 0.75$ for $\xi = 1.5$ using $H = \xi/2$.
   - **Checks:** algebraic_recompute
   - **Verdict:** PASS
   - **Notes:** Computed $H = \xi/2 = 0.75$ exactly.

6. ✔ **C6** (Page 10, Fig. 6 caption (time-lag range))

   - **Claim:** They state time lags $\tau \in [0.01, 50.0]$. Check that the interval endpoints are ordered and compute span.
   - **Checks:** range_sanity_check
   - **Verdict:** PASS
   - **Notes:** Endpoints ordered $(50.0 > 0.01)$; computed span=49.99 and ratio=5000.0.

7. ✔ **C7** (Pages 1-2 (Abstract/Intro) and throughout)

   - **Claim:** Core theoretical mapping $\alpha = 2/\xi$ implies specific α values for reported ξ; compute predicted α for $\xi = 5/3, 1.683, 1.697, 1.029, 1.5, 1.8$.
   - **Checks:** algebraic_recompute_batch
   - **Verdict:** PASS

- **Notes:** Computed $\alpha = 2/\xi$ for each ξ and verified monotonic decrease of α with increasing ξ.

8. ✔ **C8** (Page 11, Sec. 3.5 and Fig. 8 caption)

   - **Claim:** They state theoretical prediction $\alpha \approx 1.2$ derived from Eulerian roughness; check which ξ would yield $\alpha = 1.2$ via $\xi = 2/\alpha$ and consistency with earlier ξ values.
   - **Checks:** inverse_mapping_recompute
   - **Verdict:** PASS
   - **Notes:** Computed $\xi = 2/\alpha = 1.6666666666666667$; consistent with $\xi = 5/3$ and rounded 1.667 under loose tolerance for approximate $\alpha \approx 1.2$.

Limitations

- Only parsed text was available; numeric values embedded solely in figures/legends (e.g., specific measured H values in Fig. 3) cannot be extracted without image-based reading, which is out of scope per instructions.
- Many quantitative claims are outputs of regressions/fits (structure-function slopes, MLE α estimates, RG-flow $\alpha(\tau)$); without underlying data, only definitional/algebraic consistency checks are feasible.
- No tables of computed exponents are provided in the text; verification is limited to re-computations from explicitly stated numbers and formula relationships.